

# Algebras, BPS States, and Strings

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## Abstract

We clarify the role played by BPS states in the calculation of threshold corrections of D=4, N=2 heterotic string compactifications. We evaluate these corrections for some classes of compactifications and show that they are sums of logarithmic functions over the positive roots of generalized Kac-Moody algebras. Moreover, a certain limit of the formulae suggests a reformulation of heterotic string in terms of a gauge theory based on hyperbolic algebras such as  $E_{10}$ . We define a generalized Kac-Moody Lie superalgebra associated to the BPS states. Finally we discuss the relation of our results with string duality.

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## 1. Introduction

It has become clear recently that theories with extended supersymmetry have a rich dynamical structure which is nonetheless amenable to exact analysis using ideas of electromagnetic duality [1,2,3,4,5,6,7,8]. One important feature of these theories is the existence of BPS states. These states play a crucial role in the dynamics of the theory and their structure at strong coupling can in many cases be determined by a semi-classical analysis. For example, in the analysis of [5,6] certain BPS states become massless at special points in the quantum moduli space of vacua and dominate the low-energy dynamics.

In string theory there is increasing evidence that a related although undoubtedly richer structure is present, particularly in  $N = 2$  theories exhibiting duality [9,10]. String theories with extended supersymmetry resulting from toroidal compactification have in fact an infinite spectrum of BPS states [11,12]. These BPS states have played a central role in much of the recent work on duality in string theory. In Type II string theory the presence of an infinite tower of non-perturbative BPS states is crucial in understanding the strong coupling behavior and duality symmetries [13,14]. These BPS states carrying Ramond-Ramond charge are also essential to the resolution of the conifold singularity in Type II string theory [15,16] and play an important role in understanding the relation between Type I and Type II string theory [17]. In the heterotic string BPS states played a central role in the original understanding of  $S$ -duality [18]. It seems fair to say that the foundations of string theory are shifting and that the structure of BPS states provides one of the most useful clues as to what type of theory unifies these disparate phenomena.

Another theme which runs through much of string theory is the search for the symmetries which underlie the structure of string theory. One symmetry structure which has been investigated in this regard is that of hyperbolic Kac-Moody and generalized Kac-Moody algebras [19,20,21,22,23,24,25,26]. Generalized Kac-Moody (GKM) algebras occur very naturally as unbroken symmetry groups in certain string ground states [23] and -contrary to what one expects in spontaneously broken gauge theory - even when these gauge symmetries are broken they nevertheless put strong constraints on the S-matrix [24]. The idea of a  $T$ -duality invariant string algebra based on a Lorentzian lattice as a gauge algebra for  $N = 4$  heterotic string compactifications was discussed in [22] and is summarized in [27]. There are also close connections between these algebras and the structure of string field theory [28].

In this paper we will provide evidence for a connection between BPS states in string theory with  $N = 2$  spacetime supersymmetry and GKM algebras. We will show that

threshold corrections in  $N = 2$  heterotic string compactifications are in fact determined purely in terms of the spectrum of BPS states. What is more surprising is that these corrections are closely related to product formulae that have been studied recently by Borcherds, Gritsenko and Nikulin [29,30,31,32,33] in connection with generalized Kac-Moody algebras. We will show how this connection arises and construct a GKM Lie superalgebra in terms of vertex operators associated to BPS states.

Our results can be viewed as a generalization to string theory of some of the structures previously encountered in duality and  $N = 2$  Yang-Mills theory. For example, our results suggest that the BPS states in string theory should be regarded as “gauge bosons” of the GKM algebra. We find a direct generalization of the one-loop formula for the prepotential in  $N = 2$  Yang-Mills theory [7,8],

$$\mathcal{F} = \frac{i}{4\pi} \sum_{\vec{\alpha} > 0} (\alpha \cdot A)^2 \log \frac{(\alpha \cdot A)^2}{\Lambda^2} \quad (1.1)$$

where  $A$  determines the components of the Higgs expectation value in the Cartan subalgebra  $\phi = \sum A_i H_i$  and the sum in (1.1) runs over the positive roots of the Lie algebra of the gauge group  $G$ . In a certain limit we will find a similar formula in string theory where the sum over the positive roots of the Lie algebra of  $G$  is replaced by a sum over the positive roots of a GKM algebra.

The simplest example of a product formula which occurs in threshold corrections is Borcherds’ remarkable product formula for the modular  $j$  function

$$j(p) - j(q) = p^{-1} \prod_{n>0, m \in \mathbb{Z}} (1 - p^n q^m)^{c(mn)} \quad (1.2)$$

where  $j(q) - 744 = \sum_{n=-1} c(n)q^n$ . The proof can be found in [34] and [35]. The infinite product converges for  $|p|, |q| < e^{-2\pi}$  and is defined elsewhere by analytic continuation. An expansion of both sides for small  $p, q$  quickly reveals that it requires an infinite number of identities among the coefficients, the first of which is  $c(4) = c(3) + c(1)(c(1) - 1)/2$ . In [36] these identities appear as “replication” formulae involving the dimensions of irreducible representations of the monster group. In [35] the product formula (1.2) is interpreted in terms of the denominator formula for the monster Lie algebra which is an example of a GKM algebra. The quantity  $j(T) - j(U)$  has appeared in the recent literature in connection with threshold corrections to gauge couplings with  $T, U$  the moduli on a  $T^2$

in the string compactification [37,38,39].<sup>1</sup> In [37,38,39] this term was found by indirect methods. We will show here how it and various generalizations can be computed directly in terms of products which generalize (1.2). Indeed, using the integrals of appendix A it is straightforward to give an independent proof of (1.2).

There is a substantial literature on threshold corrections in  $N = 1$  and  $N = 2$  heterotic string theory [40,41,42,43,44,45,46,37,39,38,47,48]. We will make particular use of some techniques of [42] and of the result in [43][48] relating threshold corrections to the new supersymmetric index [49]. We also rely heavily on the results of [38,39] relating explicit one-loop string calculations to the quantities appearing in  $N = 2$  supergravity effective Lagrangians. A connection between BPS states and threshold corrections has been suggested, directly and indirectly, in various forms, in [50,51,47,52].

The outline of this paper is as follows. In the second section we briefly review the relevant properties of  $N = 2$  heterotic string compactifications and associated moduli spaces. In the third section we discuss the nature of perturbative BPS states in these compactifications. We show that threshold corrections are determined purely by the spectrum of BPS states and relate these corrections to the elliptic genus of the internal  $c = 6$ ,  $N = 4$  superconformal field theory governing the compactification. In the fourth section we determine the dependence of these corrections on either the  $(T, U)$  Narain moduli of  $T^2$  or these plus the  $E_8$  Wilson line moduli and show that they are naturally given by infinite sums of logarithmic functions. The fifth section contains a discussion of the relation of our work to a theorem of Borchers. We discuss the automorphic structure of the products which result from our analysis and show how Borchers results follow from an analysis of certain modular integrals. In section six we discuss the quantum monodromy of the prepotential. The seventh section contains a brief discussion of GKM algebras and their associated root lattices and compares the results of sections four and five to the denominator formula for these algebras. For the previous two choices of moduli we show that the product formulae we obtain are given by a product over positive roots of the monster Lie algebra and the hyperbolic Kac-Moody Lie algebra  $E_{10}$  respectively. We discuss two limiting cases of the prepotential in section eight, one of which suggests that the GKM algebra should be considered a gauge algebra in string theory. In the ninth section we construct a generalized Kac-Moody algebra which is in some sense associated to the BPS states. The tenth section has some preliminary remarks on the application of our results

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<sup>1</sup> We use both notations  $j(q)$  and  $j(\tau)$ , depending on context.

to  $N = 2$  string duality. In the final section we conclude and offer some speculations on possible extensions of this work. Certain modular integrals needed in the evaluation of threshold corrections are computed in an appendix.

## 2. Review of $D = 4, N = 2$ heterotic string compactifications

### 2.1. Chiral Algebra

In this paper we will be considering compactifications of the heterotic string to four spacetime dimensions with  $N = 2$  spacetime supersymmetry. There is a  $c = 9, N = 1$  internal superconformal algebra (SCA) associated to any such compactification. For such theories the spacetime supersymmetry implies that this internal SCA splits into a  $c = 6$  piece with  $N = 4$  superconformal symmetry and a  $c = 3$  piece with  $N = 2$  superconformal symmetry [53]. The  $c = 3, N = 2$  theory is constructed from two free dimension 1/2 superfields. We will indicate this decomposition of the right-moving superconformal algebra as

$$\tilde{\mathcal{A}}_{\tilde{c}=3}^{N=2} \oplus \tilde{\mathcal{A}}_{\tilde{c}=6}^{N=4} \subset \tilde{\mathcal{A}} \quad , \quad (2.1)$$

where  $\tilde{\mathcal{A}}$  is the rightmoving chiral algebra. The left-moving internal conformal field theory has  $c = 22$  but is otherwise unconstrained except by modular invariance.

The  $c = 3$  theory has a  $U(1)$  current which we denote by  $J^{(1)}$ . The  $c = 6$  theory is in general non-trivial and to be compatible with  $N = 4$  superconformal symmetry must have a level one  $SU(2)$  Kac-Moody algebra. Representations of the  $c = 6$  theory can thus be labelled by the conformal weight and  $SU(2)$  representation  $(h, I)$ . We will also choose a  $U(1)$  current  $J^{(2)}$  from this  $SU(2)$  algebra with the normalization  $J^{(2)} = 2J^3$  where  $J^3$  is the  $SU(2)$  Cartan current. The total  $U(1)$  current of the  $c = 9$  theory is  $J = J^{(1)} + J^{(2)}$ .

### 2.2. Remarks on the moduli space

$N = 2$  heterotic compactifications typically have a large moduli space of vacua. The moduli decompose into hypermultiplets and vectormultiplets under the  $N = 2$  spacetime supersymmetry. The  $c = 6, N = 4$  SCA has two massless representations in the Neveu-Schwarz sector,  $(h = 0, I = 0)$  and  $(h = 1/2, I = 1/2)$  [54]. These are associated to vectormultiplet and hypermultiplet moduli respectively. The factorization (2.1) is reflected in spacetime through the constraints of  $N = 2$  supergravity which require a local factorization of the total moduli space of the form

$$SK(n) \times Q(m) \quad (2.2)$$

where  $SK(n)$  is the vectormultiplet moduli space which must be a special Kahler manifold of real dimension  $2n$  and  $Q(m)$  is the hypermultiplet moduli space and is quaternionic of real dimension  $4m$ . In fact the moduli space is a complicated and singular space; its global structure has not been fully elucidated and should prove very interesting. The classical moduli space is an algebraic variety which probably has the following structure.  $\mathcal{M}$  is a singular stratified space, that is, there is a disjoint union

$$\mathcal{M} = \amalg_f \mathcal{M}_f \quad (2.3)$$

with smooth strata  $\mathcal{M}_f$  which fit together in a singular fashion. The  $\mathcal{M}_f$  have the structure of a fibration of a quaternionic manifold over a special Kahler manifold.<sup>2</sup>

The classical vectormultiplet moduli space for  $N = 2$  string compactifications is given by the special Kahler manifold [55]

$$\mathcal{M}_{\text{vm}}^{s+2,2} \equiv \frac{SU(1,1)}{U(1)} \times \mathcal{N}^{s+2,2} \quad (2.4)$$

where the first factor is associated with the dilaton. Here we have introduced the Narain moduli space  $\mathcal{N}^{s+2,2}$ . This is a quotient of the generalized upper half plane

$$\mathcal{H}^{s+1,1} \equiv O(s+2, 2; \mathbb{R}) / [O(s+2) \times O(2)] \quad (2.5)$$

by the left-action of the arithmetic subgroup  $O(s+2, 2; \mathbb{Z})$ :

$$\begin{aligned} \mathcal{N}^{s+2,2} &\equiv O(s+2, 2; \mathbb{Z}) \backslash \mathcal{H}^{s+1,1} \\ &= O(s+2, 2; \mathbb{Z}) \backslash O(s+2, 2; \mathbb{R}) / [O(s+2) \times O(2)] \end{aligned} \quad (2.6)$$

The special geometry of (2.6) is described in some detail below.

For  $N = 2$  compactifications with internal space  $K3 \times T^2$  and an embedding of the spin connection in  $E_8 \times E_8$  the moduli space  $\mathcal{M}_{\text{vm}}^{s+2,2}$  always contains a subvariety  $\cong \mathcal{N}^{2,2}$ , which is the Narain moduli space for the complex moduli  $T, U$  on  $T^2$ :

$$\frac{O(2,2)}{O(2) \times O(2)} \simeq \left( \frac{SU(1,1)}{U(1)} \right)_T \otimes \left( \frac{SU(1,1)}{U(1)} \right)_U. \quad (2.7)$$

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<sup>2</sup> In the case of global  $N = 2$  SYM with matter these statements can probably be proven from the  $F$  and  $D$  flatness equations. For gauge group  $G$  the strata should be enumerated by the possible unbroken subgroups  $G_1 \subset G$ . The base should be  $\mathfrak{t}(G_1) \otimes \mathbb{C} / W(G_1)$  while the fibers are hyperkahler quotients.

The arithmetic group  $O(2, 2; \mathbb{Z})$  in this case consists of the  $T \leftrightarrow U$  exchange and transformations by  $PSL(2, \mathbb{Z})_T \times PSL(2, \mathbb{Z})_U$ .

In  $N = 2$  string theory the classical special geometry of  $\mathcal{M}_{\text{vm}}^{s+2,2}$  receives quantum corrections. The dependence of threshold corrections on  $T, U$  moduli has been well studied in the literature and has played an important role in recent tests of string-string duality [56,57,58]. In many compactifications based on  $K3 \times T^2$  one finds additional vectormultiplet moduli associated to Wilson lines for the unbroken gauge group on  $T^2$ . The dependence of threshold corrections on these Wilson line moduli has been studied in [37,38,59].

In this paper we will for the most part focus on two special cases with moduli space (2.4) for  $s = 0$  and  $s = 8$  corresponding to the  $T, U$  moduli on  $T^2$  in the first case and in the second to  $T, U$  and the Wilson line moduli associated to the unbroken  $E_8$  factor for  $K3$  compactifications given by the standard embedding of the spin connection in the gauge group. For these two cases the moduli are just the Narain moduli associated to the even self-dual lattices  $\Pi^{2,2}$  and  $\Pi^{10,2}$ . However many of our considerations are more general and we will only specialize to these two cases when necessary.

### 2.3. The classical special geometry of $\mathcal{N}^{s+2,2}$ and of $\mathcal{M}_{\text{vm}}^{s+2,2}$ .

The standard Kahler geometry on the space (2.4) is special geometry described, for example, in [38,60]. We now discuss a useful parametrization of the homogeneous space

$$\mathcal{H}^{s+1,1} = \frac{O(s+2, 2)}{O(s+2) \times O(2)} \quad (2.8)$$

occurring in (2.4). Further details may be found in [61,62,38,60,37].

We can represent the homogeneous space  $\mathcal{H}^{s+1,1}$  as a “tube domain” in a complexified Minkowski space as follows. Let  $\langle \cdot, \cdot \rangle$  be a real quadratic form of signature  $(+^{s+2}, -^2)$ . Consider:

$$\{u \in \mathbb{C}^{s+4} : \langle u, u \rangle = 0, \langle u, \bar{u} \rangle < 0\} / (u \sim \lambda u) \quad (2.9)$$

This is a homogeneous space for  $O(s+2, 2; \mathbb{R})$ . We let  $y \in \mathbb{R}^{s+1,1} \otimes \mathbb{C}$ , where we have a real inner product  $(, )$  on  $\mathbb{R}^{s+1,1}$  of signature  $(+^{s+1}, -^1)$ . We parametrize the solutions to (2.9) by

$$y \rightarrow u(y) \equiv \left( y; 1, -\frac{1}{2}(y, y) \right) \quad (2.10)$$

where the inner products are related by

$$\langle v; x_1, x_2 \rangle^2 \equiv (v, v) + 2x_1 x_2 \quad (2.11)$$

Moreover

$$\langle u(y), \overline{u(y)} \rangle = +2(\Im y, \Im y) \quad (2.12)$$

so we must have  $(\Im y)^2 < 0$ , i.e., a lightlike vector <sup>3</sup>.

The lightcone has two components, take  $\Im y \in C_+$ , the forward light-cone, so we realize the moduli space as a generalized upper half plane:

$$O(s+2, 2)/[O(s+2) \times O(2)] \cong \mathcal{H}^{s+1,1} \equiv \mathbb{R}^{s+1,1} + iC_+^{s+1,1} \quad (2.13)$$

Sometimes when we want to be more specific about the inner product in (2.10) we take

$$\begin{aligned} v &= (\vec{v}; v_+, v_-) \\ (v, v) &= \vec{v}^2 - 2v_+v_- \end{aligned} \quad (2.14)$$

where  $\vec{v}^2$  is the standard Euclidean inner product.

One important special case is the submanifold  $\vec{y} = 0$ , isomorphic to the case  $s = 0$ . Then we have two complex numbers  $\Im y_+ \Im y_- > 0$ . The forward lightcone then gives

$$(y_+, y_-) \rightarrow (T, U) \in \mathcal{H} \times \mathcal{H} \quad (2.15)$$

in the product of two upper half planes.

The Kahler metric on  $\mathcal{N}^{s+2,2}$  is

$$K = -\log[-\frac{1}{2}\langle u, \bar{u} \rangle] = -\log[-(\Im y)^2] = -\log[2\Im y_+ \Im y_- - (\Im \vec{y})^2]. \quad (2.16)$$

Note that the argument of the log is positive by the definition of the domain of  $y$  and that the resulting metric is one of constant negative curvature. Similarly, the classical prepotential and Kähler potential for the space (2.4) are  $\mathcal{F} = -S(y, y)$  and  $K = -\log[8\Im S] - \log[-(\Im y)^2]$ , respectively.

For our purposes the space  $\mathcal{N}^{s+2,2}$  arises as the moduli space for Narain compactifications based on the even self-dual Lorentzian lattice  $\Pi^{8t+2,2}$  when  $s = 8t$ . <sup>4</sup> We write  $\Pi^{8t+2,2} \cong \Pi^{8t,0} \oplus \Pi^{2,2}$  and write lattice vectors as

$$(\vec{b}; m_+, n_-; m_0, n_0) \quad (2.17)$$

with metric

$$(\vec{b}; m_+, n_-; m_0, n_0)^2 = \vec{b}^2 - 2m_+n_- + 2m_0n_0. \quad (2.18)$$

Here  $\Pi^{8t,0}$  is an even self-dual Euclidean lattice. For  $t = 0$  it is the zero lattice  $\{\vec{b} = 0\}$ . It is unique for  $t = 1$ , there are two for  $t = 2$ , twenty-four for  $t = 3$ , and so forth.

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<sup>3</sup> We will use both  $\Re z$ ,  $\Im z$  and  $z_1$ ,  $z_2$  to indicate the real and imaginary parts of a complex number  $z$  in this paper

<sup>4</sup> We use the symbol  $\Pi^{p,q}$  to denote a standard even unimodular Lorentzian lattice. Lattices isomorphic to it are denoted by  $\Gamma^{p,q}(y)$ , where  $y \in \mathcal{N}^{p,q}$ .



### 3. Supersymmetric Index and BPS states in $N = 2$ heterotic string compactifications

#### 3.1. BPS States

Compactifications of the heterotic string with  $N = 2$  spacetime supersymmetry have a spacetime supersymmetry algebra which includes a complex central charge  $\mathcal{Z}$ :

$$\{Q_\alpha^i, Q_\beta^j\} = \epsilon_{\alpha\beta} \epsilon^{ij} \mathcal{Z}. \quad (3.1)$$

The central charge  $\mathcal{Z}$  is determined by the right-moving momenta  $p_R$  carried by the free superfields in the  $c = 3$ ,  $N = 2$  internal SCA. The Virasoro constraints imply that the mass of any state in this theory is given by (in the Neveu-Schwarz sector)

$$M^2 = (N_R - 1/2) + \frac{1}{2}p_R^2 + h_R \quad (3.2)$$

where  $N_R$  is the right-moving oscillator number coming from the uncompactified coordinates and the two free superfields of the  $c = 3$  theory and  $h_R$  is the conformal weight in the  $c = 6$  part of the internal SCA. As a result, bosonic BPS states which must satisfy  $M^2 = \frac{1}{2}p_R^2$  arise as right-moving ground states with either  $N_R = 1/2$  and  $h_R = 0$  or  $N_R = 0$  and  $h_R = 1/2$ .

For Narain compactifications with lattices  $\Pi^{8t+2,2}$  and lattice vectors (2.17) we have

$$\begin{aligned} \frac{1}{2}(p_L^2 - p_R^2) &= \frac{1}{2}\vec{b}^2 - m_+n_- + m_0n_0 \\ \frac{1}{2}p_R^2 &= \frac{1}{-2(\Im y)^2} \left| \vec{b} \cdot \vec{y} - m_+y_- - n_-y_+ + n_0 - \frac{1}{2}m_0y^2 \right|^2. \end{aligned} \quad (3.3)$$

That is, the central charges are determined by the inner product of the lattice vector (2.17) and the coordinate  $u(y)$ .

In contrast to theories with  $N = 4$  spacetime supersymmetry, the spectrum of BPS states in  $N = 2$  theories can have a “chaotic” nature, that is BPS states can appear and disappear under infinitesimal perturbations of the moduli. One simple example which illustrates this is a symmetric orbifold limit of  $K3$ . We consider a  $Z_2$  orbifold of a  $\Pi^{4,4}$  lattice. In the untwisted sector of the orbifold we will have states labelled by momenta  $(\tilde{p}_L, \tilde{p}_R) \in \Pi^{4,4}$ . Such states can be BPS states only if  $\tilde{p}_R = 0$ . But the existence of such states varies discontinuously as we vary the Narain moduli associated to the  $\Pi^{4,4}$  lattice. For example if we focus on one  $S^1$  factor of radius  $R$  then states with  $(\tilde{p}_L \neq 0, \tilde{p}_R = 0)$  exist only for rational values of the modulus  $2R^2$ .

This seems to be in contradiction with the usual wisdom that BPS states must behave smoothly under perturbations of the theory. The resolution of this puzzle is simply that these chaotic BPS states always appear in hypermultiplet, vectormultiplet pairs. As one moves away from the special points these BPS states pair into long representations of the  $N = 2$  spacetime supersymmetry algebra and are no longer BPS saturated. This also makes it clear that threshold corrections cannot depend only on the number and charges of BPS states if one is to obtain smooth functions of the moduli. Rather, as we will see in the following section, the threshold corrections depend only on the difference between vectormultiplet and hypermultiplet BPS states and this difference is a smooth function of the moduli.

### 3.2. BPS states and the supersymmetric index

It was shown in [43] that threshold corrections in  $N = 2$  heterotic string compactifications can be written in terms of the “new supersymmetric index” of [49]:

$$\text{Tr}_R J_0 e^{i\pi J_0} q^{L_0 - c/24} \bar{q}^{\tilde{L}_0 - \tilde{c}/24} = \frac{1}{2\pi i} \frac{\partial}{\partial \theta} \Big|_{\theta=\frac{1}{2}} \text{Tr}_{\mathcal{H}_R^{\text{int}}} q^{L_0 - c/24} \bar{q}^{\tilde{L}_0 - \tilde{c}/24} e^{2\pi i \theta J_0} \quad (3.4)$$

where the trace is over the Ramond sector of the internal  $(c, \tilde{c}) = (22, 9)$  conformal field theory and  $J_0$  is the total  $U(1)$  charge defined earlier. In the literature  $J_0$  is often denoted by  $F$ .

We will now show that for  $N = 2$  compactifications one can relate (3.4) to a sum over BPS saturated states. Using the decomposition (2.1) the Hilbert space of the internal superconformal field theory may be written as

$$\mathcal{H}^{\text{int}} = \sum_{(p_R; h, I)} \mathcal{H}_{p_R; h, I}^{(22, 0)} \otimes \tilde{\mathcal{H}}_{p_R}^{(0, 3)} \otimes \tilde{\mathcal{H}}_{h, I}^{(0, 6)} \quad (3.5)$$

where superscripts denote the Virasoro central charges  $(c, \tilde{c})$ , the second factor is a Fock space for the free  $N=2$  superfield, and the third factor is a unitary irrep of the  $N = 4$  SCA labelled by the conformal weight and  $SU(2)$  representation  $(h, I)$ . Each summand in (3.5) is a tensor product of representations of the  $\tilde{\mathcal{A}}_{N=2}^{\tilde{c}=3}$  and the  $\tilde{\mathcal{A}}_{N=4}^{\tilde{c}=6}$  algebras. Accordingly, the  $U(1)$  current may be decomposed as  $J = J^{(1)} + J^{(2)}$  and hence we may rewrite the trace on each summand in (3.5) as:

$$\begin{aligned} \text{Tr}_{p_R \otimes (h, I)} J_0 e^{i\pi J_0} q^{L_0 - c/24} \bar{q}^{\tilde{L}_0 - \tilde{c}/24} = \\ \text{Tr}_{p_R} J_0^{(1)} e^{i\pi J_0^{(1)}} \bar{q}^{\tilde{L}_0 - \tilde{c}/24} \left( \text{Tr}_{(h, I)} e^{i\pi J_0^{(2)}} q^{L_0 - c/24} \bar{q}^{\tilde{L}_0 - \tilde{c}/24} \right) \\ + \text{Tr}_{p_R} e^{i\pi J_0^{(1)}} \bar{q}^{\tilde{L}_0 - \tilde{c}/24} \left( \text{Tr}_{(h, I)} J_0^{(2)} e^{i\pi J_0^{(2)}} q^{L_0 - c/24} \bar{q}^{\tilde{L}_0 - \tilde{c}/24} \right) \end{aligned} \quad (3.6)$$

Now, for an arbitrary  $N = 4$  representation  $(h, I)$  one has

$$\text{Tr}_{(h,I)} J_0^{(2)} e^{i\pi J_0^{(2)}} q^{L_0 - c/24} \bar{q}^{\tilde{L}_0 - \tilde{c}/24} = 0. \quad (3.7)$$

To see this recall that the  $U(1)$  current is related to the  $SU(2)$  Cartan current by  $J = 2J^3$  and hence has integral spectrum. Since in  $SU(2)$  representations eigenvalues of  $J^3$  come in opposite pairs (3.7) follows. Thus, only the first term on the right hand side of (3.6) survives.

We can further simplify (3.6) using  $N=4$  representation theory. As we saw earlier BPS states correspond in the Neveu-Schwarz sector to operators in the  $N = 4$  theory with  $h_R = 0, 1/2$ . These are just the massless NS representations of the  $N = 4$  SCA with  $(h = 0, I = 0)$  and  $(h = 1/2, I = 1/2)$ . The representation  $(h = 0, I = 0)$  gives rise to the bosonic states in a BPS vectormultiplet. On the other hand the representation  $(h = 1/2, I = 1/2)$  gives rise to the bosonic states of a BPS hypermultiplet<sup>5</sup>.

Spectral flow in an  $N = 4$  theory maps these representations to Ramond representations according to

$$\begin{aligned} (h = 0, I = 0) &\rightarrow (1/4, 1/2) \\ (h = 1/2, I = 1/2) &\rightarrow (1/4, 0). \end{aligned} \quad (3.8)$$

These are the massless Ramond representations with Witten indices [54]:

$$\begin{aligned} \text{Tr}_{\mathcal{H}_{(h=1/4, I=0)}^{N=4}} (-1)^{2J_0^3} &= 1 \\ \text{Tr}_{\mathcal{H}_{(h=1/4, I=1/2)}^{N=4}} (-1)^{2J_0^3} &= -2 \end{aligned} \quad (3.9)$$

In fact, these are the only representations for which the Witten indices are nonzero [54]. It follows that the sum (3.4) reduces to a sum over BPS states.

The representation content of BPS multiplets with respect to the subalgebra  $\tilde{\mathcal{A}}_{N=2}^{c=3} \oplus \tilde{\mathcal{A}}_{N=4}^{c=6}$  may be summarized as follows. We will denote a representation of this algebra by  $(h, q) \otimes (h', I)$  where  $(h, q)$  give the conformal weight and  $U(1)$  charge of the  $c = 3, N = 2$  theory and  $(h', I)$  labels the representation of the  $N = 4$  algebra. With this understood

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<sup>5</sup> Our use of the terms “hypermultiplet” and “vectormultiplet” here is nonstandard. We define these terms based purely on the right-moving structure of the representation. For example, with this terminology the supergravity multiplet is counted as a “vectormultiplet.”

vectormultiplets and hypermultiplets have the following content in the Neveu-Schwarz (NS) and Ramond (R) sectors:

$$\begin{array}{ll}
& \text{Vectormultiplets} & \text{Hypermultiplets} \\
\text{NS :} & 2 \times (0, 0) \otimes (0, 0) \oplus (1/2, \pm 1) \otimes (0, 0) & 2 \times (0, 0) \otimes (1/2, 1/2) \\
\text{R :} & (1/8, \pm 1/2) \otimes (1/4, 1/2) & 2 \times (1/8, \pm 1/2) \otimes (1/4, 0)
\end{array} \tag{3.10}$$

where all combinations of  $\pm$  signs should be taken.

Combining (3.10) with (3.9) we see that each BPS vectormultiplet contributes

$$[\tfrac{1}{2}e^{i\pi/2} - \tfrac{1}{2}e^{-i\pi/2}](-2) = -2i \tag{3.11}$$

to  $J_0 e^{i\pi J_0}$  while each BPS hypermultiplet contributes

$$2 \times [\tfrac{1}{2}e^{i\pi/2} - \tfrac{1}{2}e^{-i\pi/2}](+1) = +2i \tag{3.12}$$

to  $J_0 e^{i\pi J_0}$ . Thus the BPS states contribute to (3.4) with vectormultiplets and hypermultiplets weighted with opposite signs, that is:

$$\begin{aligned}
& \frac{1}{\eta^2} \text{Tr}_R J_0 e^{i\pi J_0} q^{L_0 - c/24} \bar{q}^{\tilde{L}_0 - \tilde{c}/24} = \\
& -2i \left[ \sum_{\text{BPS vectormultiplets}} q^\Delta \bar{q}^{\bar{\Delta}} - \sum_{\text{BPS hypermultiplets}} q^\Delta \bar{q}^{\bar{\Delta}} \right]
\end{aligned} \tag{3.13}$$

Strictly speaking this equation is not completely correct because of the fact that non-physical states with  $\Delta \neq \bar{\Delta}$  nonetheless contribute to modular integrals. With the understanding that the contribution of these non-physical BPS states should be included as well (3.13) is a correct equation.

Note that this is in accord with the expectation that there are no threshold corrections for  $N = 4$  spacetime supersymmetry and indeed  $N = 4$  BPS states split up into a  $N = 2$  hypermultiplet and a  $N = 2$  vectormultiplet. Another way to understand this result from the spacetime point of view is to notice that when representing the extended supertranslation algebra the massive *long*  $N = 2$  representations are the same as the *short*  $N = 4$  representations. But we know there are never any threshold corrections in  $N = 4$  theories, therefore we expect threshold corrections only from short  $N = 2$  representations, that is from BPS states.

### 3.3. Elliptic genus

From (3.6) and (3.7) we see that the new supersymmetric index (3.4) depends on  $N = 4$  moduli only through

$$\text{Tr}_{\text{Ramond}} e^{i\pi J_0^{(2)} q^{L_0 - c/24} \bar{q}^{\tilde{L}_0 - \tilde{c}/24}} \quad (3.14)$$

which, roughly speaking, is the elliptic genus [63,64,65] of the  $N = 4$  conformal field theory. For a general class of backgrounds we can be more specific about the relation to the conventional elliptic genera. We assume that the gauge bundle on  $T^2 \times K3$  has the structure  $\pi_1^*(V_1) \oplus \pi_2^*(V_2)$ , that is, in the fermionic formulation of the gauge algebra, the leftmoving fermions  $\lambda^I$  can be split into two disjoint sets coupling via their currents as

$$\lambda^I A_\mu^{IJ} \partial X^\mu \lambda^J + \lambda^I B_\mu^{IJ} \partial X^\mu \lambda^J \quad (3.15)$$

where  $A$  is a flat connection on  $T^2$  for the bundle  $V_1$ , and  $B$  is an anti-self-dual instanton connection for the bundle  $V_2$  on  $K3$ . We further assume that we can embed the connection  $B$  in an  $SO(2n)$  subgroup of  $SO(16) \subset E_8$ . We can then use the bosonic formulation for one  $E_8$  factor and the fermionic formulation for the  $E_8$  factor containing the connection  $B$  and split the 16 fermions as  $16 = 2n + (16 - 2n)$ . We then couple the  $2n$  fermions to the connection  $B$  via (3.15). In this case we have  $(16 - 2n)$  free Majorana-Weyl fermions coupled to the flat connection  $A$  on  $T^2$ , and a  $(c, \tilde{c}) = (4 + n, 6)$  heterotic sigma model on  $K3$  with a gauge bundle

$$\begin{array}{c} V_2 \\ \downarrow \\ K3 \end{array} \quad (3.16)$$

satisfying the conditions

$$c_1(V_2) = 0, \quad ch_2(V_2) = \frac{1}{2}c_1(V_2)^2 - c_2(V_2) = c_2(TK3) = +24. \quad (3.17)$$

We now define elliptic genera for the  $(4 + n, 6)$  sigma-model as

$$\begin{aligned} \Phi^+(\hat{A}) &= \text{Tr}_{NS,R}(-1)^{F_R} q^{L_0 - c/24} \bar{q}^{\tilde{L}_0 - \tilde{c}/24} \\ \Phi^-(\hat{A}) &= \text{Tr}_{NS,R}(-1)^{F_L + F_R} q^{L_0 - c/24} \bar{q}^{\tilde{L}_0 - \tilde{c}/24} \\ \Phi(\Delta) &= \text{Tr}_{R,R}(-1)^{F_R} q^{L_0 - c/24} \bar{q}^{\tilde{L}_0 - \tilde{c}/24} \end{aligned} \quad (3.18)$$

where the subscript on the trace indicates the left-moving and right-moving boundary conditions respectively and  $F_L$  and  $F_R(\equiv J_0^{(2)})$  are the left and right-moving fermion number.

The  $\Phi^\pm$  are the elliptic generalization of the Dirac index [64] where the  $\pm$  determines whether odd antisymmetric tensor representations of  $SO(2n)$  are counted with a plus sign or minus sign.  $\Phi^+$  is denoted by  $H(q)$  in the first reference of [64]. The quantity  $\Phi(\Delta)$  involves the elliptic generalization of the Dirac index coupled to the spinor bundles associated to  $V_2$ .

We can combine these indices together with the remaining free left-moving fermions, summed over Ramond and Neveu-Schwarz boundary conditions to obtain for the full trace

$$\begin{aligned} & \text{Tr}_R q^{L_0 - c/24} \bar{q}^{\tilde{L}_0 - \tilde{c}/24} J_0 e^{\pi i J_0} \\ &= 2i \frac{Z_{10,2}}{\eta^{12}} \left[ \left( \frac{\vartheta_3}{\eta} \right)^{8-n} \Phi^+(\hat{A}) - \left( \frac{\vartheta_4}{\eta} \right)^{8-n} \Phi^-(\hat{A}) + \left( \frac{\vartheta_2}{\eta} \right)^{8-n} \Phi(\Delta) \right] \end{aligned} \quad (3.19)$$

where  $Z_{10,2}/\eta^{12}(\tau)$  is the partition function for the free bosonic degrees of freedom on the  $\Pi^{10,2}$  lattice constructed from the  $E_8$  lattice and the  $\Pi^{2,2}$  lattice of  $T^2$ .

Since the elliptic genus depends only on the topology of the manifold and the topology of the gauge bundle it is invariant under deformations of the hypermultiplet moduli in accord with the decompositions (2.2) and (2.1). Thus it can be evaluated by working in a special limit of the hypermultiplet moduli such as an orbifold limit. For the standard embedding of the spin connection in an  $SU(2)$  subgroup of  $E_8$  we can simply combine (3.19) with the results of [66] to obtain the answer.

## 4. Threshold corrections and the Prepotential

### 4.1. General strategy

In this section and the following we establish a connection between threshold corrections in  $N = 2$  compactifications of heterotic string theory and product formulae studied recently by Borcherds in connection with generalized Kac-Moody algebras. This strongly suggests the presence of a GKM algebra in such compactifications. In the ninth section we will discuss a construction of this GKM algebra in terms of vertex operators for BPS states.

We will determine the prepotential by comparing two formulae for one-loop renormalizations of nonabelian gauge couplings<sup>6</sup>.

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<sup>6</sup> In this section we follow the “string-theoretic” conventions of [48][38] for moduli. Thus, for example  $\Re S > 0$ ,  $\Re T > 0$ , etc. The conventions which are useful for discussing automorphic properties are related by

$$y^{\text{string}} = -iy^{\text{automorphic}}$$

For a gauge group  $G$  the one-loop coupling renormalization is given by [48]:

$$\begin{aligned} \frac{1}{g^2(G; p^2)} &= k\Re\left[S + \frac{1}{16\pi^2}\Delta^{\text{univ}}\right] + \frac{b(G)}{16\pi^2}\log\frac{M_{\text{string}}^2}{p^2} + \frac{1}{16\pi^2}\Delta(G; y) \\ \Delta(G; y) &= \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \left[ \mathcal{B} - b(G) \right] \end{aligned} \quad (4.1)$$

where  $\mathcal{B}$  is given by a trace over the internal Hilbert space:

$$\mathcal{B} = -\frac{i}{\eta^2} \text{Tr} \mathcal{H}_{\text{R}}^{\text{int}} \left\{ J_0 e^{i\pi J_0} q^{L_0 - 22/24} \bar{q}^{\bar{L}_0 - 9/24} \left[ Q^2 - \frac{k}{8\pi\tau_2} \right] \right\} \quad (4.2)$$

Here  $Q$  is a generator of the gauge group,  $b(G)$  is the coefficient of the one-loop beta function normalized as in [48] and  $S$  is the dilaton field. The “universal” term  $\Delta^{\text{univ}}$  is related to the “Green-Schwarz” term which governs the one-loop mixing of the axion and moduli fields and plays a role in the cancellation of sigma-model anomalies [67,68,48]. The quantity  $k$  is the level of the Kac-Moody algebra associated to  $G$ . Henceforth we set  $k = 1$ .

From the discussion in the previous section it is clear that (4.2) receives contributions only from BPS states. Morally speaking

$$\frac{1}{g^2} \sim \frac{1}{16\pi^2} \left[ \sum_{vm} 2Q^2 \log m^2 - \sum_{hm} 2Q^2 \log m^2 \right] \quad (4.3)$$

in accord with the threshold rule of [38] eqn. 3.45.

We can further elaborate (4.1) using the constraints of  $N = 2$  supergravity following the work of [38]. In (4.1) we may express the universal threshold correction in terms of the one-loop correction to the prepotential  $h^{(1)}$ :

$$\frac{1}{16\pi^2} \Delta^{\text{univ}} = \frac{1}{-(\Re y)^2} \Re \left[ h^{(1)} - y_1^a \frac{\partial}{\partial y^a} h^{(1)} \right] \quad (4.4)$$

On the other hand, using the Wilsonian coupling we may also write [48][38]:

$$\frac{1}{g^2(G; p^2)} = \Re \left[ \tilde{S} - \frac{1}{(s+4)\pi^2} \log[\Psi(y)] \right] + \frac{b(G)}{16\pi^2} \left[ \log \frac{M_{\text{Planck}}^2}{p^2} + K(S, \bar{S}, y, \bar{y}) \right] \quad (4.5)$$

where

$$K = -\log[\Re(S)] - \log[-(y_1)^2] + \text{const.} \quad (4.6)$$

and a subscript 1 denotes the real part. The Planck mass is given by  $M_{\text{Planck}}^2 = M_{\text{string}}^2 \Re S$ . Since we are using the string theory conventions of [48,38] for the moduli the real part of  $y$

is in the forward lightcone. The coupling  $\tilde{S}$  differs from  $S$  by the addition of a holomorphic function such that  $\tilde{S}$  is invariant under the duality group up to shifts. In [38] it is shown that we may write <sup>7</sup>:

$$\tilde{S} = S + \frac{1}{s+4} \eta^{ab} \frac{\partial}{\partial y^a} \frac{\partial}{\partial y^b} h^{(1)} \quad (4.7)$$

and under the perturbative duality group we have

$$\tilde{S} \rightarrow \tilde{S} + \frac{i}{2(s+4)} \eta^{IJ} \Lambda_{IJ} \quad (4.8)$$

where  $\Lambda_{IJ}$  is a real symmetric matrix.

In (4.5)  $\Psi(y)$  is a holomorphic function on  $\mathcal{H}^{s+1,1}$  such that

$$\Re \left[ \frac{1}{(s+4)\pi^2} \log[\Psi(y)] + \frac{b(G)}{16\pi^2} \log[-(y_1)^2] \right] \quad (4.9)$$

is invariant under the perturbative duality group  $O(s+2, 2; \mathbb{Z})$ . Requiring that the physical coupling be free of any singularity on  $\mathcal{N}^{s+2,2}$  fixes the divisor of  $\Psi$ . These conditions determine  $\Psi(y)$  up to an overall constant, since if we have two such then  $\Psi_1/\Psi_2$  is a well-defined automorphic function extending to the compactification divisors of the variety  $\mathcal{N}^{s+2,2}$  and is thus a constant. <sup>8</sup>

Equating (4.1) and (4.5) gives a formula for the prepotential. We now present the solution for some special cases. We will consider a compactification of the heterotic string on  $K3 \times T^2$  with the standard embedding of the spin connection in the gauge group. This breaks the  $E_8 \times E_8$  gauge group to  $E_7 \times E_8$ . In the further reduction to four dimensions on  $T^2$  one can choose Wilson lines for the remaining unbroken subgroup leading at generic points to a low-energy gauge group  $U(1)^{s+4}$  with  $s = 15$  with the  $U(1)^4$  arising from the two left-moving and two right-moving  $U(1)$ 's associated with the  $T^2$  factor and the remaining  $U(1)^{15}$  corresponding to the Cartan subalgebra of  $E_7 \times E_8$ .

When the unbroken gauge group is  $[E_8 \times E_7 \times U(1)^2]_{left} \times [U(1)^2]_{right}$  (or an enhancement thereof) we will compute the running  $E_8$  gauge coupling:

$$\frac{1}{g^2(E_8; p^2)} \quad s = 0 \quad (4.10)$$

---

<sup>7</sup> Note the change of sign from 4.30 in [38]. We introduce  $\tilde{S}$  so that we can also discuss the (10, 2) case. In the (2, 2) case it is related to the invariant coupling  $S^{\text{inv}}$  of [38] via:  $\tilde{S} = S^{\text{inv}} - L/8$ .

<sup>8</sup> Strictly speaking one must specify appropriate asymptotic conditions to guarantee uniqueness. However we do not need the precise statement since we obtain the prepotential by direct computation.



as a function of the moduli  $T, U$  parametrizing  $\mathcal{N}^{2,2}$ . When the gauge group is  $[E_7 \times U(1)^{10}]_{left} \times [U(1)^2]_{right}$  we will compute

$$\frac{1}{g^2(E_7; p^2)} \quad s = 8 \quad (4.11)$$

as a function of the moduli in  $\mathcal{N}^{10,2}$ . The calculation for both cases can be done in parallel. The answer is given in section 4.4 below.

#### 4.2. Computing the integrand

The first step in the calculation is the evaluation of the integrand in (4.1). As discussed in the previous section, the hypermultiplet dependence of the integrand enters only through the elliptic genus and therefore depends only on the topology of the gauge bundle and the manifold specifying the compactification. Thus, although the integrand in principle requires evaluating a partition function of a super conformal field theory on K3, we can perform the computation in an orbifold or other limit which is smoothly connected to the theory of interest. We will evaluate corrections by working in the  $T^4/Z_2$  orbifold limit of K3 with the standard orbifold embedding of the twist in the gauge group. Related computations have been done for the elliptic genus of K3 in [66].

We start with the Narain lattice for a  $T^6$  compactification to four dimensions with the decomposition

$$\Gamma^{22,6} = \Gamma^{10,2} \oplus \Gamma^{4,4} \oplus \Gamma^{8,0} \quad (4.12)$$

We then mod out by an involution which acts as  $-1$  on the  $\Gamma^{4,4}$  factor and as a shift  $X^I \rightarrow X^I + \delta^I$  on  $\Gamma^{8,0}$  with

$$\delta = (\tfrac{1}{2}, \tfrac{1}{2}, 0^6). \quad (4.13)$$

For this particular orbifold limit the unbroken gauge group at generic points is in fact  $U(1)^{20}$  and the vectormultiplet moduli space is  $\mathcal{N}^{18,2}$ . However we will restrict ourselves in what follows to studying the dependence of the threshold corrections on the moduli of the  $\Gamma^{10,2}$  factor in (4.12) for which the classical moduli space is  $\mathcal{N}^{10,2}$ . At least for orbifolds it should be possible to study the dependence on other moduli by generalizing the formulae in the appendix to congruence subgroups of  $SL(2, Z)$ .

We thus can evaluate:

$$\begin{aligned}
\eta^{-2}(\tau) \text{Tr}_{\mathbf{R}\mathbf{q}}^{L_0-c/24} \bar{\mathbf{q}}^{\tilde{L}_0-\tilde{c}/24} \mathbf{J}_0 e^{\pi i \mathbf{J}_0} &= -2i Z_{10,2} \frac{E_6}{\eta^{24}} \\
\eta^{-2}(\tau) \text{Tr}_{\mathbf{R}\mathbf{q}}^{L_0-c/24} \bar{\mathbf{q}}^{\tilde{L}_0-\tilde{c}/24} \mathbf{J}_0 e^{\pi i \mathbf{J}_0} [\mathbf{Q}^2(E_7) - \frac{1}{8\pi\tau_2}] &= -\frac{i}{12} Z_{10,2} \mathcal{E}_7 \\
\eta^{-2}(\tau) \text{Tr}_{\mathbf{R}\mathbf{q}}^{L_0-c/24} \bar{\mathbf{q}}^{\tilde{L}_0-\tilde{c}/24} \mathbf{J}_0 e^{\pi i \mathbf{J}_0} [\mathbf{Q}^2(E_8) - \frac{1}{8\pi\tau_2}] &= -\frac{i}{12} Z_{2,2} \mathcal{E}_8
\end{aligned} \tag{4.14}$$

where

$$\begin{aligned}
Z_{s+2,2} &= \sum_{p \in \Gamma^{s+2,2}} q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}p_R^2} \\
\mathcal{E}_7 &= \left( \frac{(E_2 - \frac{3}{\pi\tau_2})E_6 - E_4^2}{\eta^{24}} \right) \\
\mathcal{E}_8 &= \left( \frac{(E_2 - \frac{3}{\pi\tau_2})E_4E_6 - E_6^2}{\eta^{24}} \right)
\end{aligned} \tag{4.15}$$

Here  $E_{2n}(\tau)$  are the usual Eisenstein series and are given explicitly in the appendix. As a check note that the first equation in (4.14) follows from (3.19) and the results of [66].

It follows that we can rewrite the coupling in (4.1) as

$$\begin{aligned}
\frac{1}{g^2(G; p^2)} &= \Re \left[ S + \frac{1}{16\pi^2} \Delta^{\text{univ}} \right] + \frac{b(G)}{16\pi^2} \log \frac{M_{\text{string}}^2}{p^2} \\
&\quad - \frac{1}{12} \frac{1}{16\pi^2} \left( \tilde{\mathcal{I}}_{s+2,2} - \mathcal{I}_{s+2,2} \right)
\end{aligned} \tag{4.16}$$

Here we have introduced a class of integrals defined in equations (A.1), (A.2) of appendix A. The specific integrals appearing in (4.16) involve the modular forms:

$$\begin{aligned}
s = 8 : \quad \frac{E_6}{\eta^{24}} &= \sum c_1(n) q^n = q^{-1} - 480 + \dots \\
\frac{E_4^2}{\eta^{24}} &= \sum c_3(n) q^n = q^{-1} + 504 + \dots
\end{aligned} \tag{4.17}$$

$$\begin{aligned}
s = 0 : \quad \frac{E_6 E_4}{\eta^{24}} &= \sum c_1(n) q^n = q^{-1} - 240 + \dots \\
\frac{E_6^2}{\eta^{24}} &= \sum c_3(n) q^n = q^{-1} - 984 + \dots
\end{aligned} \tag{4.18}$$

Note that the subtraction of the constant terms appearing in the definition of the integrals  $\tilde{\mathcal{I}}, \mathcal{I}$  and the beta function are consistent for

$$\begin{aligned}
b(G) &= -\frac{1}{12} (\tilde{c}_1(0) - c_3(0)) \\
&= -60 \quad s = 0 \\
&= +84 \quad s = 8
\end{aligned} \tag{4.19}$$

In general the coefficients  $c_1(n)$  and  $c_3(n)$  depend both on the choice of gauge bundle and on which low-energy gauge group is being studied.

#### 4.3. Formulae for the Integrals

The integrals  $\mathcal{I}, \tilde{\mathcal{I}}$  are evaluated in appendix A. The integral  $\mathcal{I}$  is given by the expression:

$$\begin{aligned}\mathcal{I}_{s+2,2}(y) &= -2 \log |\Phi(y)|^2 + c_3(0) \left( -\log[-(\Re y)^2] - \mathcal{K} \right) \\ \Phi(y) &= e^{-2\pi\rho \cdot y} \prod_{r>0} \left( 1 - e^{-2\pi r \cdot y} \right)^{c_3(-r^2/2)}\end{aligned}\tag{4.20}$$

In the above  $\mathcal{K}$  is a constant defined in the appendix. The product over  $r > 0$  means the following. We consider the even-self-dual Lorentzian lattice  $\Pi^{s+1,1}$  for  $s = 0, 8$  and write lattice vectors as  $r = (\vec{b}, -\ell, -k)$  with  $\vec{b} \in \Pi^{s,0}$ . For  $s = 8$ ,  $\Pi^{8,0}$  is the root lattice of  $E_8$  and we choose a set of positive roots for this lattice. With this understood,  $r > 0$  means:

1.  $k > 0$  or,
2.  $k = 0, \ell > 0$ , or,
3.  $k = \ell = 0, \vec{b} > 0$ .

For  $s = 8$  we show in section seven below that this is the positive root condition for  $E_{10}$ . For  $s = 0$  the coefficient  $c_3(0)$  is not determined purely by modular invariance. After explicitly dividing out the terms involving  $c_3(0)$  the remaining product is over the positive roots of the Monster Lie algebra [34][35].

The vector  $\rho$  is:

$$\begin{aligned}\rho &= \rho_{E_{10}} = -(\vec{\rho}; 31, 30) \quad s = 8 \\ \rho &= -\frac{c_3(0)}{24}(1, 1) + (0, 1) \quad s = 0.\end{aligned}\tag{4.21}$$

where  $\vec{\rho}$  is the Weyl vector of  $E_8$ . For  $s = 8$ ,  $\rho_{E_{10}}$  is the Weyl vector for  $E_{10}$ . For  $s = 0$  (4.21) gives a lattice Weyl vector for the  $\Pi^{1,1}$  lattice but only gives the Weyl vector for the Monster Lie algebra after subtraction of the  $c_3(0)$  term.

The second integral we need is<sup>9</sup>:

$$\begin{aligned}\tilde{\mathcal{I}}_{s+2,2}(y) &= 4\Re \left\{ \sum_{r>0} \left[ \tilde{c}_1\left(-\frac{r^2}{2}\right) Li_1(e^{-2\pi r \cdot y}) + \frac{6}{\pi(y_1)^2} c_1\left(-\frac{r^2}{2}\right) \mathcal{P}(ir \cdot y) \right] \right\} \\ &\quad + \tilde{c}_1(0) \left( -\log[-(y_1)^2] - \mathcal{K} \right) + \frac{1}{(y_1)^2} [\tilde{d}_{ABC}^{s+2,2} y_1^A y_1^B y_1^C + \delta]\end{aligned}\tag{4.22}$$

---

<sup>9</sup> Integrals of this type are also evaluated in [69] although there are important differences in our results.

where

$$\delta = \frac{6}{\pi^2} c_1(0) \zeta(3). \quad (4.23)$$

Here  $Li_1(x) = -\log(1-x)$  and the function  $\mathcal{P}$  involves the polylogarithms  $Li_2$  and  $Li_3$  and is defined in the appendix. As explained in section 5.1,  $\mathcal{H}^{s+1,1}$  is tessellated by a system of Weyl chambers, and the symmetric tensor  $\tilde{d}^{s+2,2}$  is piecewise constant in  $\mathcal{H}^{s+1,1}$ , taking different values in different Weyl chambers. Explicit formulae for it are given in appendix A.

#### 4.4. Answer for the prepotential

Equating (4.16) with (4.5) and using (4.4) and (4.7) we obtain a differential equation for  $h^{(1)}$ :

$$\begin{aligned} \Re \left[ \frac{1}{s+4} \eta^{ab} \frac{\partial}{\partial y^a} \frac{\partial}{\partial y^b} h^{(1)} \right] + \frac{1}{(\Re y)^2} \Re \left[ h^{(1)} - y_1^a \frac{\partial}{\partial y^a} h^{(1)} \right] \\ = -\frac{1}{12} \frac{1}{16\pi^2} \left( \tilde{\mathcal{I}}_{s+2,2} - \mathcal{I}_{s+2,2} \right) + \frac{1}{(s+4)\pi^2} \Re[\log \Psi] + \frac{b(G)}{16\pi^2} \log(-y_1^2). \end{aligned} \quad (4.24)$$

A particular solution of the differential equation (4.24), *in the fundamental Weyl chamber*,<sup>10</sup> is given by

$$h^{(1)} = \frac{1}{384\pi^2} \tilde{d}_{ABC}^{s+2,2} y^A y^B y^C - \frac{1}{2(2\pi)^4} c_1(0) \zeta(3) - \frac{1}{(2\pi)^4} \sum_{r>0} c_1(-r^2/2) Li_3(e^{-2\pi r \cdot y}) \quad (4.25)$$

Substitution into (4.24) leads to a solution with

$$\log \Psi(y) = \frac{1}{2} \log[J(iT) - J(iU)] + b(E_8) \log[\eta(iT)\eta(iU)] \quad s = 0 \quad (4.26)$$

and

$$\Psi(y) = \Phi(y) = e^{-2\pi \rho \cdot y} \prod_{r>0} \left( 1 - e^{-2\pi r \cdot y} \right)^{c_3(-r^2/2)} \quad s = 8. \quad (4.27)$$

*Proof.* We substitute directly the ansatz for  $h^{(1)}$ . The universal term is given by

$$\frac{1}{16\pi^2} \Delta^{\text{univ}} = \frac{1}{y_1^2} \frac{1}{(2\pi)^3} \Re \left[ \sum_{r>0} c_1(-r^2/2) \mathcal{P}(ir \cdot y) \right] + \frac{1}{192\pi^2 (y_1)^2} [\tilde{d}_{ABC}^{s+2,2} y_1^A y_1^B y_1^C + \delta] \quad (4.28)$$

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<sup>10</sup> See section 5.1

In order to cancel the other terms we must use the identity:

$$\begin{aligned}\sum n c_1(n) q^n &= -\frac{1}{2} \frac{E_2 E_6 + E_4^2}{\eta^{24}} & s = 8 \\ \sum n c_1(n) q^n &= -\frac{1}{6} \frac{E_2 E_4 E_6 + 2E_6^2 + 3E_4^3}{\eta^{24}} & s = 0\end{aligned}\tag{4.29}$$

One can check the rational terms directly, but a more elegant method relates the trace of  $\tilde{d}_{ABC}$  to the Weyl vector. For example, in the case  $s = 8$  we proceed as follows. The Laplacian for the Kahler metric on  $\mathcal{N}^{s+2,2}$  is given by

$$\nabla^2 = -2y_1^2 \left( \eta^{ab} - \frac{2}{y_1^2} y_1^a y_1^b \right) \partial_a \bar{\partial}_b \tag{4.30}$$

By straightforward computation one finds:

$$\begin{aligned}(\nabla^2 - 24)\tilde{\mathcal{I}}_{10,2} + 280\tilde{c}_1(0) &= 24 \left[ -2 \log |\Phi(y)|^2 \right. \\ &\quad \left. - c_3(0)(\log(-y_1^2) + \mathcal{K}) + (8\pi\rho_b - \frac{1}{4}\tilde{d}_{ab}^a)y_1^b \right]\end{aligned}\tag{4.31}$$

since both sides of the equation must be invariant under the duality group we have after comparing to (4.20):

$$\tilde{d}_{ab}^a = -32\pi\rho_b \tag{4.32}$$

Similarly one finds

$$\begin{aligned}(\nabla^2 - 4)\tilde{\mathcal{I}}_{2,2} &= 4\tilde{c}(0) \left( \log[2T_1 U_1 |\eta(iT)^4 \eta(iU)^4|] + \mathcal{K} - \frac{1}{2} \right) \\ &\quad - 80 \log |J(iT) - J(iU)|\end{aligned}\tag{4.33}$$

Using this equation one can check that the rational terms in (4.25) solve the equation required for equality of (4.1) and (4.5).

Above we have exhibited a particular solution of the second order differential equation. Two solutions to this equation must differ by a solution of the homogeneous equation. It is straightforward to show that the only solutions of the homogeneous equation which are analytic around zero are of the form:

$$\delta h^{(1)} = \sum_{0 \leq m+n \leq 2} a_{IJ,mn} (\hat{X}^I)^m (\hat{X}^J)^n \tag{4.34}$$

where the coefficients  $a_{IJ,mn}$  are pure imaginary and <sup>11</sup>

$$\hat{X}^I = (1, -y^2, iy^a). \quad (4.35)$$

Two prepotentials differing by such an expression are physically equivalent. We may use this result, together with the automorphic properties discussed in section six to argue that the prepotential given above is the unique answer.

It is worthwhile to make several checks on the above answer. First, the physical coupling must be a nonsingular function on moduli space. We can check this since the singularities of  $\log \Psi$  are identical to  $\frac{1}{s+4} \left( \frac{\partial}{\partial y} \right)^2 h^{(1)}$ . Second, the physical coupling must be duality invariant. This is a consequence of the automorphic properties proved in the next two sections. Third, in the case  $s = 0$ , a differential equation for the quantity  $S^{\text{inv}} = S - \frac{1}{16\pi^2} \Delta^{\text{univ}}$  was derived in [48,38]. <sup>12</sup> It is straightforward to show that the above formula for  $h^{(1)}$  satisfies this constraint. As a fourth check we may compare the “Yukawa couplings” <sup>13</sup> following from (4.25) with those derived in [38][39]. For example, one finds, in the  $s = 0$  case:

$$\partial_U^3 h^{(1)} = -\frac{1}{2\pi} \left[ 1 - \sum_{r>0} c_1(k\ell) \ell^3 \frac{e^{-2\pi(kT+\ell U)}}{1 - e^{-2\pi(kT+\ell U)}} \right] \quad (4.36)$$

According to [38,39] (4.36) must coincide with

$$-\frac{1}{2\pi} \frac{E_4(iU)E_4(iT)E_6(iT)}{(J(iT) - J(iU))\eta^{24}(iT)} \quad (4.37)$$

Comparing the  $T \rightarrow \infty$  and  $U \rightarrow \infty$  limits of the expressions one finds perfect agreement. Expanding in power series and comparing terms we find agreement to  $10^{th}$  order.

#### 4.5. Gravitational Corrections

There are other terms in the effective supergravity action which involve chiral densities and are therefore computable to all orders of perturbation theory. The first of these is the gravitational coupling  $F_1$  given by

$$F_1 = \frac{-i}{192\pi^2} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \left[ \frac{1}{\eta^2} \text{Tr}_R J_0(-1)^{J_0} q^{L_0-22/24} \bar{q}^{\tilde{L}_0-9/24} \left[ E_2 - \frac{3}{\pi\tau_2} \right] - b_{\text{grav}} \right] \quad (4.38)$$

---

<sup>11</sup> in a gauge where  $X^0 = 1$

<sup>12</sup> See, for example, equation 4.46 in [38].

<sup>13</sup> Actually, magnetic moments, [60].

where the  $E_2$  factor arises from the  $Q_{grav}^2 = -2\partial_\tau \log \eta$  term in [43]. In heterotic string theory  $F_1$  and its generalizations  $F_g$  can be computed at one-loop order and the comparison between these calculations and genus  $g$  computations in dual twisted Calabi-Yau theories provide strong evidence for  $N = 2$  string duality [58,56]. Again it is clear from the previous discussion that these quantities in heterotic string theory receive contributions only from BPS states. There are also potential ambiguities in these amplitudes which require a careful treatment of infrared divergences, perhaps along the lines presented in [69]<sup>14</sup>.

## 5. Comment on relation to work of Borchers

In this section we will discuss the automorphic properties of the threshold corrections we have computed and compare our results to work of Borchers. We will work in “automorphic” conventions for the moduli as described in footnote 4 of the previous section.

First we explain the relation between Borchers’ “rational quadratic divisors” (RQD) and enhanced symmetry points (ESP) in Narain compactifications. Let  $L \cong \Pi^{s+1,1}$  be an even unimodular lattice, and consider the  $\Pi^{s+2,2}$  lattice  $M \cong L \oplus \Pi^{1,1}$ . We represent an element of  $M$  by  $v = (r; a_+, a_-)$  with  $r \in L$  and  $a_\pm$  integer. In [29] Borchers defines rational quadratic divisors to be the locus  $\mathcal{D}(v) \subset \mathcal{H}^{s+1,1}$  of

$$\langle (r; -a_+, a_-), (y; 1, -\frac{1}{2}(y, y)) \rangle = 0 \quad (5.1)$$

for

$$\langle (r; -a_+, a_-), (r; -a_+, a_-) \rangle = r^2 - 2a_+a_- > 0 \quad (5.2)$$

From the discussion in sec. 3.1 we know that BPS states are parametrized by vectors  $v = (r; a_+, a_-) \in \Pi^{s+2,2} \cong \Pi^{s+1,1} \oplus \Pi^{1,1}$ . The central charge of a BPS state with quantum numbers  $(r; a_+, a_-) \in \Pi^{s+1,1} \oplus \Pi^{1,1}$  is just the inner product [37]

$$\mathcal{Z}(v, y) = (r, y) + a_- - \frac{1}{2}a_+y^2 = \langle (r; a_+, a_-), (y; 1, -\frac{1}{2}y^2) \rangle \quad (5.3)$$

Moreover, for such a state

$$p_L^2 - p_R^2 = r^2 + 2a_+a_- \quad (5.4)$$

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<sup>14</sup> We thank J. Louis for pointing this out to us.

Thus in string theory only the divisors with  $\langle (r; -a_+, a_-), (r; -a_+, a_-) \rangle = 2$  are of importance and these correspond to enhanced symmetry points in the Narain moduli space.

In [29][30] Borcherds has proved the following theorem:

**Theorem.** Let  $f(\tau) = \sum c(n)q^n$  be a meromorphic modular form with all poles at cusps. Suppose that  $f$  is of weight  $-s/2$  for  $SL(2, \mathbb{Z})$  and has integer coefficients, with  $24|c(0)$  if  $s = 0$ . Then there is a unique vector  $\rho \in L$  such that

$$\Phi(y) = e^{2\pi i \rho \cdot y} \prod_{r>0, r \in \Pi^{s+1,1}} (1 - e^{2\pi i r \cdot y})^{c(-r \cdot r/2)} \quad (5.5)$$

can be analytically continued to define a meromorphic automorphic form of weight  $c(0)/2$  for  $O(s+2, 2; \mathbb{Z})^+$ . All the zeroes and poles of  $\Phi$  lie on rational quadratic divisors, and the multiplicity of the zero of  $\Phi$  at the rational quadratic divisor of the triple  $(b; a_1, a_2)$  is

$$\sum_{n>0} c(n^2(a_1 a_2 - (b, b)/2)) \quad (5.6)$$

In (5.5)  $r > 0$  means that  $r$  has positive inner product with a chosen negative norm vector in  $L$ . In some cases the product (5.5) is known to be the denominator formula for a generalized Kac-Moody algebra.

We have seen that products of precisely this type arise in the analysis of threshold corrections in  $N = 2$  heterotic string compactifications. Indeed, by studying the properties of the integrals  $\mathcal{I}_{s+2,2}(y)$  of appendix A we can easily rederive many of the results discovered by Borcherds.

To do this we must explain the holomorphic factorization of the product arising in the expression for  $\mathcal{I}$  in equation (A.29) of appendix A. In the previous section we presented the results only in a particular Weyl chamber (defined below). However the integrals as evaluated in the appendix depend on the “hatted” dot product  $r \hat{\cdot} y$  rather than the dot product  $r \cdot y$ . The quantity  $r \hat{\cdot} y$  is defined by:

$$r \hat{\cdot} y \equiv \Re \left[ (\vec{b} \cdot \vec{y} + \ell y_- + k y_+) \right] + i \left| \Im \left[ (\vec{b} \cdot \vec{y} + \ell y_- + k y_+) \right] \right| \quad (5.7)$$

when  $k > 0$ , and, when  $k = 0$ :

$$r \hat{\cdot} y \equiv r \cdot y - N(r, y) y_- \quad (5.8)$$

where

$$N(r, y) = \text{sgn}(\vec{b}) \left[ \text{sgn}(\vec{b}) \frac{\vec{b} \cdot \Im \vec{y}}{y_{-,2}} \right] \quad (5.9)$$

is an integer ( $[\cdot]$  is the greatest integer function.) In order to understand the holomorphic factorization we must introduce Weyl chambers.



### 5.1. Weyl chambers

We can tessellate the forward lightcone  $C_+^{s+1,1} \subset \mathcal{H}^{s+1,1}$  into convex polyhedra whose walls are defined by the *real* codimension one subvarieties

$$\Im(r \cdot y) = 0 \quad (5.10)$$

for  $r^2 = 2, r \in \Pi^{s+1,1}$ . These are the “surfaces of marginal stability” and must be crossed when circling the divisors of vanishing BPS central charges.

Given a choice of simple roots  $r_\mu$  for the lattice  $\Pi^{s+1,1}$  we can define a *fundamental Weyl chamber* by the equation

$$\mathcal{CW}_0 \equiv \{y : r_\mu \cdot \Im y > 0\} \quad (5.11)$$

Explicit choices of simple roots are described in section seven. For  $s = 0$  there are two Weyl chambers and we take the fundamental Weyl chamber to be simply  $y_{+,2} > y_{-,2}$ . For  $s = 8$  we choose a set of positive roots  $\vec{b} > 0, \vec{b}^2 = 2$  for  $E_8$ . Using the simple roots in the next section we get the conditions:

$$\begin{aligned} 0 < \frac{\vec{b} \cdot \Im \vec{y}}{y_{-,2}} < 1 \\ y_{-,2} < y_{+,2} \end{aligned} \quad (5.12)$$

### 5.2. Holomorphic factorization and $\mathcal{I}_{s+2,2}(y)$

By definition of the fundamental Weyl chamber we know that  $r \cdot y = r \cdot \Im y$  for the positive roots. Thus, the products (A.31), (A.37) factorize straightforwardly in the fundamental chamber. The question arises as to the other chambers.

Contributions from  $c(n)$  for  $n > 0$  always holomorphically factorize. To see this note that  $\Im y$  is in the forward light cone so

$$\begin{aligned} \Im y_+ &> 0 \\ \Im y_- &> 0 \\ 2\Im y_+ \Im y_- &> (\Im \vec{y})^2 \end{aligned} \quad (5.13)$$

Since we are studying coefficients  $c(n)$  for  $n > 0$  in the product we have  $2k\ell > (\vec{b})^2$ . It follows that

$$(k\Im y_+ + \ell\Im y_-)^2 > 4(k\Im y_+)(\ell\Im y_-) > |\vec{b} \cdot \Im \vec{y}|^2 \quad (5.14)$$

so the imaginary part of  $r \cdot y$  is always positive and the product is a holomorphic square. Similarly, the coefficient  $c(0)$  enters for roots  $k = 0, \ell > 0, \vec{b} = 0$  and for roots  $k\ell = \frac{1}{2}\vec{b}^2 > 0$ . For these roots we can again drop the hat. All of this is quite simply understood from the physical point of view: only states with  $v^2 = 2$  can become massless and lead to nonanalytic behavior.

The real source of nontrivial holomorphic factorization is entirely in the coefficients of  $c(-1)$  which are connected to RQD's or ESP's. For example, the peculiar shift in (5.9) affects these terms. The key observation is that these changes can be absorbed in a change of the linear term  $\rho \cdot y \rightarrow \rho_\alpha \cdot y$  in the RHS of (A.29) in each Weyl chamber  $\mathcal{C}_\alpha$ . Consider for example, the holomorphic factorization for roots with  $r^2 = +2$  and  $k > 0$ . Then:

$$\log|1 - e^{2\pi i r \cdot y}|^2 = \begin{cases} \log|1 - e^{2\pi i r \cdot y}|^2 & \text{for } \Im r \cdot y > 0 \\ \log|e^{-2\pi i r \cdot y}(1 - e^{2\pi i r \cdot y})|^2 & \text{for } \Im r \cdot y < 0 \end{cases} \quad (5.15)$$

For example, consider the case  $s = 0$ . There is only one positive root with  $r^2 = 2$ , namely  $r = (\ell, k) = (-1, 1)$ . The wall is just

$$\Im y_+ = \Im y_- \quad (5.16)$$

i.e.  $T_2 = U_2$ . Consequently the product in (4.20) becomes

$$-2 \log \left| e^{-2\pi i T c(-1)} \prod_{r>0} \left( 1 - e^{2\pi i (kT + \ell U)} \right)^{c(k\ell)} \right|^2 \quad (5.17)$$

for  $T_2 > U_2$  and

$$-2 \log \left| e^{-2\pi i U c(-1)} \prod_{r>0} \left( 1 - e^{2\pi i (kU + \ell T)} \right)^{c(k\ell)} \right|^2 \quad (5.18)$$

for  $U_2 > T_2$ .

### 5.3. Automorphic properties of $\Phi$

We will now use invariance of the integral under the group generated by

1.  $y \rightarrow y + \lambda$ ,  $\lambda \in \Pi^{s+1,1}$
2.  $y \rightarrow w(y)$ ,  $w \in O(s+1, 1; \mathbb{Z})^+$
3.  $y \rightarrow y' = 2y/(y, y)$

to deduce some automorphic properties of the “holomorphic square root”  $\Phi(y)$ . It is evident that the first two transformations act on  $\mathcal{H}^{s+1,1}$ . The third is surprising, but note that under this transformation:

$$(\Im y', \Im y') = + \frac{4}{|(y, y)|^2} (\Im y, \Im y) \quad (5.19)$$

and so preserves the “upper half plane.” In fact, the transformations 1., 2. and 3. only generate an index 2 subgroup of the duality group, and  $\Phi$  is automorphic for the entire group  $O(s+2, 2; \mathbb{Z})$ .

Invariance under  $y \rightarrow y + \lambda$  is trivial. To verify  $y \rightarrow w(y)$  invariance we use (5.15) which says that in a Weyl chamber  $\mathcal{C}_\alpha$  we have

$$\mathcal{I} = -2 \log |e^{(\rho_\alpha - \rho) \cdot y} \Phi(y)|^2 + c(0) \left( -\log [-(\Im y)^2] - \mathcal{K} \right) \quad (5.20)$$

Now we note that

$$(\rho_\alpha - \rho) = w(\rho) - \rho \quad (5.21)$$

To prove this note that it is true for chambers related to the fundamental chamber by a reflection in a simple root  $r_i$ . Then proceed from there by induction.

The transformation of  $\Phi$  under the third generator is the most difficult to prove by straightforward methods. We may note that if  $\Re y = 0$  then the transformation is a rescaling of  $\Im y$  by a positive coefficient. Therefore, an open set of the fundamental Weyl chamber is mapped to itself. From the invariance of the integral  $\mathcal{I}_{s+2,2}$  and its holomorphic factorization we then find the modular transformation law:

$$\Phi(y') = \left[ \frac{(y, y)}{2} \right]^{c(0)/2} \Phi(y) \quad (5.22)$$

#### 5.4. Zeroes and Poles of $\Phi$

Finally, we come to the last part of Borchers’ theorem which is concerned with the zeros and poles of  $\Phi$ . Unlike the walls of the Weyl chambers, which are determined by  $r \in \Pi^{s+1,1}$ , the RQD’s are determined by vectors  $v \in \Pi^{s+2,2}$  with  $v^2 = 2$ . As  $y \rightarrow \mathcal{D}(v)$  for some vectors in the Narain lattice  $\Gamma^{s+2,2}(y)$ ,  $p_R \rightarrow 0$ ,  $p_L^2 \rightarrow 2$ . Each such vector contributes a logarithmic divergence to  $\mathcal{I}_{r+2,2}(y)$ :

$$\int^\infty \frac{d\tau_2}{\tau_2} e^{-4\pi\tau_2 \frac{1}{2} p_R^2} \rightarrow -\log |\mathcal{Z}(v, y)|^2 \quad (5.23)$$

*Analysis of RQD's: (2,2) case*

The RQD's in  $\mathcal{H}^{1,1}$  are given by the divisor  $T = U$  and modular transformations of it. These have a double intersection at the points  $T = U = i$  and a triple intersection at  $T = U = \rho$  (and their modular images). This is in accord with standard analysis of the enhanced symmetry points.

One proof of the above statements proceeds by writing:

$$\mathcal{Z} = \begin{pmatrix} y_+ & 1 \end{pmatrix} \begin{pmatrix} n_0 & n_- \\ m_+ & m_0 \end{pmatrix} \begin{pmatrix} y_- \\ 1 \end{pmatrix} \quad (5.24)$$

where  $r^2 = 2$  implies  $m_0 n_0 - m_+ n_- = 1$ . Note that  $SL(2, Z) \times SL(2, Z)$  acts on the sets of divisors. By acting on  $y_-$  we can transform the matrix to the unit matrix, so the divisor equation becomes  $1 + y_+ y_- = 0$  or, by a further  $SL(2, \mathbb{Z})$  transformation  $y_+ = y_-$ . We obtain the set of all divisors in  $\mathcal{H} \times \mathcal{H}$  by taking  $SL(2, Z) \times SL(2, Z)$  images of this divisor. Note that divisors can only intersect when both  $y_+$  and  $y_-$  are fixed by some element of  $SL(2, \mathbb{Z})$ . In particular, the divisor  $y_+ = y_-$  has a double intersection at  $i$  and a triple intersection at  $\rho$ .

*Analysis of RQD's: (10,2) case*

It would be interesting to give a classification of the RQD's in  $\mathcal{H}^{9,1}$  analogous to that above. An important class of RQD's are defined by the simple roots of  $E_8$ . For example, define the hyperplane

$$\mathcal{H}_i = \{y : \vec{y} \cdot \vec{\alpha}_i = 0\} \quad (5.25)$$

Along this hyperplane  $\Phi$  has a simple zero

$$\Phi(y) \rightarrow -2\pi i \vec{\alpha}_i \cdot \vec{y} \Phi'(y') \quad (5.26)$$

where, for  $y' \in \mathcal{H}_i$  we have

$$\Phi'(y') = e^{2\pi i \rho' \cdot y'} \prod_{r' > 0} (1 - e^{2\pi i r' \cdot y'})^{c'_3(-r'^2/2)} \quad (5.27)$$

where  $r' \sim r$  if  $r - r' \propto \alpha_i$  and  $c'_3(-r'^2/2) = \sum_{r' \sim r} c_3(-(r')^2/2)$ . Restricting to successive intersections of divisors we get a series of automorphic forms beginning with the  $E_{10}$  form  $\Phi$  of (4.20) and ending with  $J(T) - J(U)$ .

## 6. Quantum Monodromy

### 6.1. General Remarks

The nature of the semiclassical monodromy of  $N = 2$  heterotic string compactifications has been thoroughly analyzed in references [38,39]. Using the above explicit expressions we can compute the monodromy directly.

As discussed in [60,38,39] the best basis of special coordinates for questions of monodromy is the basis  $\hat{X}^I$  related to the basis  $S, y^a$  with prepotential  $\mathcal{F} = (X^0)^2 S(y, y)$  by a duality transformation  $\hat{X}^1 = F_1, \hat{F}_1 = -X^1$ . Explicitly we have:

$$\begin{aligned}\hat{X}^1 &= X^0(y, y)^2 \\ \hat{X}^a &= X^0 y^a \quad a = 2, \dots, s+3\end{aligned}\tag{6.1}$$

In this basis the general transformation rule for the one-loop prepotential, in automorphic conventions is:

$$h^{(1)}(\tilde{y}) = \left(\frac{\tilde{X}^0}{X^0}\right)^{-2} \left[ h^{(1)}(y) - i\Lambda_{IJ} \frac{\hat{X}^I}{X^0} \frac{\hat{X}^J}{X^0} \right]\tag{6.2}$$

where  $\Lambda_{IJ}$  is a real symmetric matrix. Thus, the prepotential is an automorphic form of weight  $-2$  transforming with shifts.

In discussing monodromy one should always bear in mind that the prepotential is ambiguous by an addition of a term of the form (4.34).

### 6.2. Monodromies for $s = 0$

It is convenient to introduce the function

$$\begin{aligned}\mathcal{L}(T, U) &\equiv \sum_{r>0} c(-r^2/2) Li_3(e^{2\pi i(kT+\ell U)}) \\ &= Li_3(e^{2\pi i(T-U)}) + \sum_{k,\ell \geq 0} c(k\ell) Li_3(e^{2\pi i(kT+\ell U)})\end{aligned}\tag{6.3}$$

where

$$F(q) = \sum_{n=-1}^{\infty} c(n) q^n = \frac{E_4 E_6}{\eta^{24}} \quad .\tag{6.4}$$

This function has a branch locus at  $T = U$  and can be defined by power series for  $T_2 > U_2$ , i.e., in the fundamental Weyl chamber.

The function  $Li_3$  can be analytically continued outside the unit circle and satisfies the connection formula [70]:

$$Li_3(e^x) = Li_3(e^{-x}) + \frac{\pi^2}{3}x - \frac{i\pi}{2}x^2 - \frac{1}{6}x^3\tag{6.5}$$

Thus, under analytic continuation into the other Weyl chamber with  $U_2 > T_2$  we have:

$$\mathcal{L}(T, U) = \mathcal{L}(U, T) - (2\pi)^4 \frac{i}{24\pi} [2U^3 - 2T^3 - 3(T - U)^2 + (T - U)(6TU - 1)] \quad (6.6)$$

Under the generators  $T \rightarrow T + 1$  and  $U \rightarrow U + 1$  of the duality group the function  $\mathcal{L}(T, U)$  is invariant in its region of convergence. Finally, to compute the monodromy under  $T \rightarrow -1/T$  (and similarly for  $U \rightarrow -1/U$ ) we rewrite the integral  $\tilde{\mathcal{I}}_{2,2}$  as:

$$\begin{aligned} \tilde{\mathcal{I}}_{2,2} = & -\frac{6}{U_2 T_2 \pi^2} \Re \left[ \left( 1 - iU_2 \frac{\partial}{\partial U} \right) \left( 1 - iT_2 \frac{\partial}{\partial T} \right) \mathcal{L}(T, U) \right] \\ & + 20 \log |J(T) - J(U)| + 264 [\log [2T_2 U_2 |\eta(T)\eta(U)|^4] - \mathcal{K}] - \frac{\delta}{2T_2 U_2} \\ & + 16\pi \frac{U_2^2}{T_2} \quad T_2 \geq U_2 \\ & + 16\pi \frac{T_2^2}{U_2} \quad U_2 \geq T_2 \end{aligned} \quad (6.7)$$

We know the integral  $\tilde{\mathcal{I}}_{2,2}$  is invariant under  $T \rightarrow T' = -1/T, U \rightarrow U$  and from this is it straightforward to derive the transformation law:

$$\begin{aligned} \mathcal{L}(T', U) &= T^{-2} \left[ \mathcal{L}(T, U) + \mathcal{R}(T, U) \right] \\ \mathcal{R}(T, U) &= \frac{\pi^2}{12} \delta(1 - T^2) - \frac{4\pi^3 i}{3} U^3 (T^2 - 1) \quad T'_2 > U_2 \\ &= \frac{\pi^2}{12} \delta(1 - T^2) - \frac{4\pi^3 i}{3} \left( \frac{1 - T^4}{T} \right) + \frac{4\pi^3 i}{3} U^3 \quad T'_2 < U_2 \end{aligned} \quad (6.8)$$

In the 2,2 case there are two Weyl chambers, and accordingly we have the two expressions for the prepotential (in automorphic conventions):

$$\begin{aligned} h^{(1)} &= -\frac{1}{(2\pi)^4} \mathcal{L}(T, U) - \frac{\delta}{192\pi^2} - i \frac{U^3}{12\pi} + \delta h_+^{(1)} \quad T_2 > U_2 \\ &= -\frac{1}{(2\pi)^4} \mathcal{L}(U, T) - \frac{\delta}{192\pi^2} - i \frac{T^3}{12\pi} + \delta h_-^{(1)} \quad T_2 < U_2 \end{aligned} \quad (6.9)$$

where  $\delta h_{\pm}^{(1)}$  correspond to the ambiguity (4.34).

Using these expressions and the above transformation rules for  $\mathcal{L}(T, U)$  it is straightforward to compute the monodromy of  $h^{(1)}$ .

### 6.3. Remarks on the $s = 8$ case

In principle the above discussion generalizes in a fairly straightforward way to the computation of the monodromy on  $\mathcal{N}^{10,2}$ . The Weyl reflections  $\sigma_i$  in the  $E_{10}$  roots no longer square to zero because of the monodromy associated with the  $Li_3$  terms around enhanced symmetry varieties. The transformation of  $\mathcal{L}(y)$  under the inversion  $y \rightarrow 2y/(y, y)$  can be deduced from an analog of (6.7) for  $\tilde{\mathcal{I}}_{10,2}$ . We hope to return to a detailed discussion of this monodromy representation, which gives an interesting representation of the  $E_{10}$  braid group, in the future.

## 7. GKM algebras and denominator formulae

In this section we will provide a very brief summary of some of the properties of GKM algebras and the hyperbolic Kac-Moody algebra  $E_{10}$  which enter into the interpretation of the product formulae for threshold corrections found in the previous section. For further details on Kac-Moody algebras and GKM algebras the reader can consult [34,71,72,73,25].  $E_{10}$  is discussed in [74,75,26,76,28]

We first give the formal definition in terms of Cartan matrices, generators and relations. Recall the usual definition of a finite-dimensional simple Lie algebra. One starts with a symmetric  $r \times r$  Cartan matrix  $A = (a_{ij})$  with  $i, j = 1, 2, \dots, r$  satisfying

$$\begin{aligned} a_{ii} &= 2, \\ a_{ij} &\leq 0 \quad \text{for } i \neq j, \\ a_{ij} &\in \mathbb{Z}, \end{aligned} \tag{7.1}$$

and  $\det A > 0$ . The Lie algebra is then defined in terms of generators  $(e_i, f_i, h_i)$  obeying the relations

$$\begin{aligned} [h_i, h_j] &= 0, \\ [h_i, e_j] &= a_{ji} e_j, \\ [h_i, f_j] &= -a_{ji} f_j, \\ [e_i, f_j] &= \delta_{ij} h_j \\ (\text{ad}_{e_i})^{1-a_{ji}} e_j &= 0, \\ (\text{ad}_{f_i})^{1-a_{ji}} f_j &= 0. \end{aligned} \tag{7.2}$$

A formal construction of Kac-Moody Lie algebras is obtained by weakening the condition that  $\det A > 0$ . In particular one obtains affine Lie algebras by allowing  $A$  to have a zero

eigenvalue and hyperbolic Kac-Moody algebras by allowing a single negative eigenvalue. For Kac-Moody algebras one has the usual notion of root spaces, positive roots, simple roots, and the Weyl group generated by reflections in the real simple roots. Furthermore there is a denominator formula

$$e^\rho \prod_{r>0} (1 - e^r)^{\text{mult}(r)} = \sum_{\sigma \in W} (\text{sgn}(\sigma)) e^{\sigma(\rho)} \quad (7.3)$$

where  $\rho$  is the Weyl vector, the product on the left hand side runs over all positive roots and each term is weighted by the root multiplicity  $\text{mult}(r)$ , and on the right hand side the sum is over all elements of the Weyl group  $W$ . For example for the  $su(2)$  level one affine Lie algebra (7.3) is just the Jacobi triple product identity.

Generalized Kac-Moody algebras have a great deal in common with Kac-Moody algebras of hyperbolic type but differ from them in that simple roots  $r$  with  $r^2 \leq 0$  are allowed. The formal definition follows from a slight generalization of the conditions (7.1). One again starts with a symmetric Cartan matrix  $A = (a_{ij})$  which is allowed to have infinite rank in general. Then one demands the conditions

$$\begin{aligned} a_{ii} &= 2 \quad \text{or} \quad a_{ii} \leq 0, \\ a_{ij} &\leq 0 \quad \text{for} \quad i \neq j, \\ a_{ij} &\in \mathbb{Z} \quad \text{if} \quad a_{ii} = 2, \end{aligned} \quad (7.4)$$

and defines the algebra by generators  $(h_{ij}, e_i, f_i)$  and relations

$$\begin{aligned} [h_{ij}, e_k] &= \delta_{ij} a_{jk} e_k, \\ [h_{ij}, f_k] &= -\delta_{ij} a_{jk} f_k, \\ [e_i, f_j] &= h_{ij} \\ (\text{ad}_{e_i})^{1-a_{ji}} e^j &= 0, \quad (\text{ad}_{f_i})^{1-a_{ji}} f^j = 0, \quad a_{ii} = 2, i \neq j, \\ [e_i, e_j] &= 0, \quad [f_i, f_j] = 0, \quad a_{ii} \leq 0, a_{jj} < 0, a_{ij} = 0. \end{aligned} \quad (7.5)$$

The fact that  $a_{ii} \leq 0$  is allowed shows that imaginary simple roots ( $r^2 \leq 0$ ) appear in contrast to Kac-Moody algebras. There is again a denominator formula for GKM algebras which reads

$$e^\rho \prod_{r>0} (1 - e^r)^{\text{mult}(r)} = \sum_{\sigma \in W} (\text{sgn}(\sigma)) \sigma(e^\rho \sum_r \epsilon(r) e^r) \quad (7.6)$$

where the correction factor on the right involves  $\epsilon(r)$  which is  $(-1)^n$  if  $r$  is the sum of  $n$  distinct pairwise orthogonal imaginary roots and zero otherwise.



As with hyperbolic Kac-Moody algebras, these formal definitions are of little practical utility without some method of determining the root multiplicities. For a few special GKM algebras it is possible in some cases to determine root multiplicities through the use of product formulae.

A more useful construction of a GKM algebra arises as the vertex operator algebra of physical string states in compactifications of string theory based on Lorentzian lattices, for example in toroidal compactifications of all string coordinates including time [20,77,23,25]. In our application the metric is the Narain metric rather than the spacetime metric but similar mathematical considerations apply.

We now discuss two specific examples related to the product representation of threshold corrections obtained in the previous section.

### 7.1. $\Pi^{1,1}$

We will first consider a GKM algebra with root lattice  $\Pi^{1,1}$ . Lattice vectors are labelled by a pair of integers  $(m, n)$  and the inner product of two lattice vectors is  $(m, n) \cdot (m', n') = -(m'n + n'm)$ . There is a single real simple root of length squared two which can be chosen to be  $r_{-1} = (1, -1)$ .

The Weyl group is generated by reflections in the real simple roots and so here is just the  $Z_2$  transformation acting as

$$\sigma_{-1}(r) = r - (r \cdot r_{-1})r_{-1} \quad (7.7)$$

and thus takes  $(m, n) \rightarrow (n, m)$ .

A lattice Weyl vector is defined to be a lattice vector  $\rho$  which satisfies [78]

$$\rho \cdot r = -r^2/2 \quad (7.8)$$

for any simple real root. Applying this to  $r = (1, -1)$  shows that lattice Weyl vectors must have the form  $\rho = (m, m+1)$  for some integer  $m$ .

We now compare this to the Weyl vector  $\rho$  appearing in (4.21)

$$\rho = c(-1)(0, 1) - \frac{c(0)}{24}(1, 1) \quad (7.9)$$

where by the conditions given  $F(q) = \sum_{n=-1}^{\infty} c(n)q^n$  is a modular function (that is weight zero). Thus  $F(q) = (j(q) - 744) + c(0)$ . Taking  $c(-1) = 1$  and using in addition the condition that  $24|c(0)$  we see that (7.9) gives a lattice Weyl vector.

Borcherds has investigated two different GKM algebras related to the  $\Pi^{1,1}$  lattice. The first is the “fake” monster Lie algebra based on the Lie algebra of physical states on the lattice  $\Pi^{25,1} = \Lambda_{\text{Leech}} \oplus \Pi^{1,1}$ . This algebra is just the algebra of physical string states in a covariant theory with the transverse degrees of freedom of the string compactified on the Leech lattice and the remaining space and time dimensions on the  $\Pi^{1,1}$  lattice. The second theory is the monster Lie algebra based roughly speaking on the Lie algebra of physical states associated with the tensor product of the  $\Pi^{1,1}$  system with the monster vertex algebra of FLM based on a  $Z_2$  orbifold of the Leech lattice [79]. In this case  $c(0) = 0$  since all the massless transverse states are projected out and the Weyl vector is  $\rho = (0, 1)$ . In (4.20) we have a product over roots  $r = (-\ell, -k)$  satisfying the conditions  $k > 0$ ,  $\ell \in Z$  or  $k = 0$ ,  $\ell > 0$ . Since  $c(0) = 0$  the only roots of the algebra (as compared to the lattice) appearing in this product are those with  $k > 0$ ,  $\ell \in Z$  (and with  $\ell \geq -1$ ).

We will now show that the condition  $r > 0$  encountered in the threshold corrections is the positive root condition for a GKM algebra with Weyl vector  $\rho = (0, 1)$ . According to a theorem in [71] if  $r$  is a positive root then  $r^2 \leq -2\rho \cdot r$  with equality if and only if  $r$  is simple. With  $r = (-\ell, -k)$  and  $\rho = (0, 1)$  as above this gives the inequality

$$-2\ell k \leq -2\ell \tag{7.10}$$

which is satisfied for the roots  $r = (-\ell, -k)$  with  $k > 0$ ,  $\ell \in Z$  and furthermore this shows that the simple roots are the real root  $(1, -1)$  and the imaginary roots  $(-\ell, -1)$  for  $\ell > 0$ .

Using the facts that the unique real simple root has multiplicity one and that there are no pairwise orthogonal simple roots the denominator formula (7.6) gives

$$p^{-1} \prod_{\ell \in Z, k > 0} (1 - p^k q^\ell)^{\text{mult}(-\ell, -k)} = \sum_{\ell \geq -1, n \neq 0} \text{mult}(-\ell, -1) (p^\ell - q^\ell) \tag{7.11}$$

with  $p = e^{-(0,1)}$  and  $q = e^{-(1,0)}$ . As was shown by Borcherds [35] the undetermined root multiplicities can be determined indirectly based on the denominator formula (1.2). For Borcherds this root multiplicity arises in the construction of the monster Lie algebra. This leads to root multiplicities of  $\text{mult}(-\ell, -k) = c(k\ell)$  in (7.11) where the coefficients  $c(n)$  appear in the expansion of the modular  $j$  function with constant term set equal to zero,  $j(q) - 744 = \sum c(n)q^n$ . Here we obtain similar products but the root multiplicity is now interpreted as the ( $Z_2$  graded) multiplicity of BPS states. The formula (7.11) is precisely that appearing in (4.20) up to the  $c_3(0)$  term after substituting the characters  $p = e^{2\pi iT}$  and  $q = e^{2\pi iU}$ .

For the compactification constructed previously we have  $c(0) = -984$  which leads to the Weyl vector  $\rho = (41, 42)$ . This does not seem to lead to a simple positive root condition for a GKM algebra. However the  $\Pi^{1,1}$  case is rather exceptional in that the coefficient  $c(0)$  is not determined by the conditions on  $F(q)$ . There is therefore some ambiguity in the precise interpretation of the product formula we have obtained related to the treatment of the  $c(0)$  terms. If we explicitly evaluate these terms in the product then the remaining product is as described above a product over the positive roots of the monster Lie algebra. This ambiguity does not arise for lattices  $\Pi^{s+1,1}$  with  $s > 0$ . We now turn to such an example.

## 7.2. $\Pi^{9,1}$

In this subsection we show that the condition  $r > 0$  appearing in the threshold corrections is identical to the positive root condition for  $E_{10}$ . Recall that the condition  $r > 0$  for  $r = (\vec{b}; -, -\ell, -k)$  with  $s = 8$  means:

$$k\ell - \frac{1}{2}\vec{b}^2 \geq -1 \quad \text{and,} \quad (7.12a)$$

$$k > 0 \quad \text{or,} \quad (7.12b)$$

$$k = 0, \quad \ell > 0 \quad \text{or,} \quad (7.12c)$$

$$k = \ell = 0, \quad \vec{b} > 0 \quad . \quad (7.12d)$$

where we have taken into account that the multiplicities  $c(-r^2/2)$  vanish for  $r^2 > 2$  in (7.12a).

On the other hand, a choice of simple roots for the  $E_{10}$  lattice is given by taking:

$$r_i = (\vec{b}; 0, 0) = (\vec{\alpha}^{(i)}; 0, 0) \quad i = 1, 8 \quad (7.13)$$

where  $\vec{\alpha}^{(i)}$  is a set of simple roots of  $E_8$ , together with the extra root of  $E_9$ ,

$$r_0 = (-\vec{\theta}; -1, 0), \quad (7.14)$$

( $\vec{\theta}$  is the highest root of  $E_8$ ), and

$$r_{-1} = (\vec{0}; 1, -1). \quad (7.15)$$

Given a set of simple roots  $r_\mu$  the positive roots of  $E_{10}$  are simply

$$r = a_{-1}r_{-1} + a_0r_0 + a_i r_i \quad (7.16)$$

with positive  $a_\mu$ . We claim this set coincides with the set  $r > 0$  above.

Several cases need to be checked. The only one that causes any difficulty is the case  $k > 0, \ell > 0$ . To check this suppose

$$k\ell \geq \frac{1}{2}b^2 - 1. \quad (7.17)$$

Let

$$a_i \vec{\alpha}_i = \vec{b} + (k + \ell) \vec{\theta} \quad (7.18)$$

We then want to show that  $a_i > 0$ .

Proof: Let  $\lambda_i$  be the fundamental weights dual to  $\alpha_i$ . Then

$$a_i = \vec{\lambda}_i \cdot \vec{b} + (k + \ell) \vec{\lambda}_i \cdot \vec{\theta}. \quad (7.19)$$

Now, by the Schwarz inequality and (7.17) we have

$$|\vec{\lambda}_i \cdot \vec{b}| \leq \sqrt{2k\ell + 2} |\lambda_i| \quad (7.20)$$

so

$$a_i \geq -\sqrt{2k\ell + 2} |\lambda_i| + (k + \ell) \vec{\lambda}_i \cdot \vec{\theta} \quad (7.21)$$

Now  $\vec{\lambda}_i \cdot \vec{\theta} > 0$ , and for  $k > 0, \ell > 0$  we have:

$$(k + \ell) \vec{\lambda}_i \cdot \vec{\theta} \geq \sqrt{2k\ell + 2} |\lambda_i| \quad (7.22)$$

if we satisfy

$$(\vec{\lambda}_i \cdot \vec{\theta})^2 \geq \vec{\lambda}_i^2 \quad (7.23)$$

which is equivalent to

$$n_i^2 \geq G_{ii} \quad (7.24)$$

with  $G_{ii}$  the diagonal elements of the quadratic form of  $E_8$  and  $n_i$  the numerical marks (or Coxeter labels) of  $E_8$ . This is true by inspection.

Moreover, the Weyl vector  $\rho = (-\vec{\rho}, -31, -30)$  can be identified with the  $E_{10}$  lattice Weyl vector. To show this it suffices to check

$$\rho \cdot r_\mu = -\frac{1}{2} r_\mu^2 \quad \mu = -1, 0, 1, \dots, 8 \quad (7.25)$$

As a nontrivial check on the above statements one can prove that the vectors which satisfy the inequality:

$$(r, r) \leq -2(\rho, r) \quad (7.26)$$

for all positive roots of a GKM algebra mentioned above are precisely the vectors satisfying (7.12a, b, c, d) above.

We thus conclude that the product formula we obtain through fundamental domain integrals which in turn determine the threshold corrections can be written in terms of a product over the positive roots of  $E_{10}$ . Furthermore, the product formula determines a (graded) multiplicity associated to the roots of  $E_{10}$ . The precise relation between these multiplicities and the root multiplicities of the  $E_{10}$  hyperbolic Kac-Moody algebra is not evident to us.

## 8. Some Limiting Cases

Further insight into the meaning of the formula for the prepotential can be gleaned from examining some limiting cases.

### 8.1. $M_{\text{string}} \rightarrow \infty$

In this limit we expect to recover the standard formulae of global N=2 supersymmetry. On the moduli space  $\mathcal{N}^{10,2}$  we consider a generic point  $y_0 = (\vec{0}; y_+, y_-)$  on the embedded  $\mathcal{N}^{2,2}$  submanifold. In the limit  $M_{\text{string}} \rightarrow \infty$  a fiber of the normal bundle of this submanifold is identified with the vectormultiplet moduli space of the global  $E_8$  theory. Working near  $\mathcal{H}^{1,1} \subset \mathcal{H}^{9,1}$  we let

$$y = y_0 + (\delta\vec{y}; 0, 0) \quad (8.1)$$

and in terms of dimensionful fields we let

$$\delta\vec{y} = \frac{\vec{a}}{M_{\text{string}}} \quad (8.2)$$

In the global limit the constraint  $2y_{+,2}y_{-,2} > (\delta\vec{y}_2)^2$  is trivially satisfied for fixed  $\vec{a}$  and hence, after modding out by the duality group we have a coordinate  $[\vec{a}] \in \underline{\mathfrak{t}}(E_8) \otimes \mathbb{C}/W$ , as expected.

Restoring string units, using (4.25), and the  $N = 2$  nonrenormalization theorems, the prepotential to all orders of perturbation theory is given by:

$$\mathcal{F} = M_{\text{string}}^2 \left[ -Sy^2 - \frac{1}{(2\pi)^4} \sum_{r>0} c_1(-r^2/2) Li_3(e^{2\pi i r \cdot y}) + \mathcal{A}(y) \right] \quad (8.3)$$

where  $\mathcal{A}(y)$  is a cubic polynomial. Now, in taking the  $M_{\text{string}} \rightarrow \infty$  limit (with  $\vec{a}$  held fixed) we may extract the global prepotential from

$$\mathcal{F} \rightarrow M_{\text{string}}^2 Sy_0^2 + i\mathcal{F}^{\text{global}} + \mathcal{O}\left(\frac{|\vec{a}|}{M_{\text{string}}}\right) \quad (8.4)$$

In order to take the limit we note that the positive roots  $r = (\vec{r}; -\ell, -k) > 0$  with  $k \neq 0$  or  $\ell \neq 0$  come in pairs with  $\pm \vec{r}$ . Using this and the limiting formula

$$Li_3(1-x) \rightarrow -\frac{1}{2}x^2 \log x + \mathcal{O}(x^3 \log x) \quad (8.5)$$

for the  $k = \ell = 0$  roots we obtain a sum only over the positive roots of  $E_8$ :

$$i\mathcal{F}^{\text{global}} = \tilde{S}\vec{a}^2 - \frac{1}{8\pi^2} \sum_{\vec{\alpha}>0} (\vec{\alpha} \cdot \vec{a})^2 \log \left[ \frac{2\pi i \vec{\alpha} \cdot \vec{a}}{M_{\text{string}}} \right] \quad (8.6)$$

in agreement with the standard one-loop prepotential of the global theory, where, for any group  $G$  we have the one-loop expressions [7,8]

$$\begin{aligned} \mathcal{F} &= \frac{i}{4\pi} \sum_{\vec{\alpha}>0} (\alpha \cdot A)^2 \log \frac{(\alpha \cdot A)^2}{\Lambda^2} \\ K &= 2\Re \left[ i\mathcal{F}_i \overline{A^i} \right] \\ &= -\frac{1}{\pi} \Re \left[ \sum_{\vec{\alpha}>0} |\alpha \cdot A|^2 \left( \log \frac{(\alpha \cdot A)^2}{\Lambda^2} + 1 \right) \right] \end{aligned} \quad (8.7)$$

In (8.6) the renormalized “classical” coupling is

$$\begin{aligned} \tilde{S} &= S + \frac{1}{8\pi^2} \sum_{r>0} c_1(-r^2/2) d(\vec{r}^2) \\ &\quad \left[ e^{2\pi i r \cdot y_0} Li_2(e^{2\pi i r \cdot y_0}) + e^{4\pi i r \cdot y_0} Li_1(e^{2\pi i r \cdot y_0}) \right] \end{aligned} \quad (8.8)$$

the quantity  $d(n)$  is defined by

$$\sum_{\vec{r}^2=n} (\vec{r} \cdot \vec{a})^2 = d(n) \vec{a}^2. \quad (8.9)$$

Finally, the cubic terms in  $\mathcal{A}(y)$  vanish. The ambiguous quadratic terms with imaginary coefficients correspond to a change in scale of  $M_{\text{string}}$ .

## 8.2. $M_{\text{string}} \rightarrow 0$

One could also try taking the opposite,  $M_{\text{string}} \rightarrow 0$  limit. This raises many new conceptual questions <sup>15</sup> but, through the analogy with spontaneously broken gauge theory, might be expected to reveal fundamental underlying symmetries in string theory [80,23,24].

We must now introduce normalized fields for the other two moduli

$$y_{\pm} = \frac{a_{\pm}}{M_{\text{string}}} \quad (8.10)$$

In the low energy theory the fields  $a_{\pm}$  may be viewed as Higgs fields in the Cartan subalgebra of the enhanced  $SU(2) \times SU(2)$  or  $SU(3)$  gauge symmetry. Thus  $M_{\text{string}} \rightarrow 0$  holding  $a$  fixed is equivalent to  $y \rightarrow \infty$ . More fundamentally  $a_{\pm}$  may be viewed as geometrical data on  $T^2$ . Consider the two-torus with vanishing  $B$  field and a diagonal metric  $G_{11} = R_1^2$  and  $G_{22} = R_2^2$ . Then  $T = 2iR_1R_2$  and  $U = iR_2/R_1$ . Thus the  $y \rightarrow \infty$  limit is equivalent to a decompactification to *five* dimensions. In this limit the dominant terms in  $\mathcal{F}$  are the cubic terms in  $\mathcal{A}(y)$  <sup>16</sup>.

By the automorphic properties of  $\mathcal{F}$  the limit  $y \rightarrow \infty$  is equivalent to a limit with  $y \rightarrow 0$ . Taking a formal  $y \rightarrow 0$  limit leads us to the following suggestive formulae. We formally take the  $y \rightarrow 0$  limit term-by-term using (8.5) and

$$\mathcal{P}(x) \rightarrow \pi x \bar{x} \log x + \mathcal{O}(x^3 \log x). \quad (8.11)$$

Applying this to the formulae for the prepotential (8.3) we get:

$$f^{\text{perturbative}} \rightarrow -S(a)^2 - \frac{1}{8\pi^2} \sum_{r>0} c(-r^2/2)(r \cdot a)^2 \log\left[\frac{2\pi i r \cdot a}{M_{\text{string}}}\right] \quad (8.12)$$

and the Kahler potential becomes, in the  $y \rightarrow 0$  limit:

$$K \rightarrow -\log \left[ (\Im S)(-\Im a)^2 - \frac{1}{16\pi^2} \Re \left[ \sum_{r>0} c(-r^2/2) |r \cdot a|^2 \log\left[\frac{2\pi i r \cdot a}{M_{\text{string}}}\right]^2 \right] \right] \quad (8.13)$$

If we furthermore take a special weak-coupling limit with  $(\Im S)(-\Im a)^2 \rightarrow \infty$  then we recover the form of the global answer (8.7) but with the sum running over the positive roots of  $E_{10}$ .

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<sup>15</sup> and perhaps should not be attempted without the full nonperturbative answer

<sup>16</sup> This cubic polynomial is probably related to the very special geometry of  $D = 5, N = 2$  supergravity [81]

The limits (8.12)(8.13) are formal since the sums do not converge. Nevertheless comparison with the formulae for the global case suggests that the weak coupling 5-dimensional theory may be viewed, from the four-dimensional theory as an  $N = 2$  gauge theory for a generalized Kac-Moody algebra - most of which is spontaneously broken- with the full, infinite, spectrum of BPS states playing the role of a tower of gauge bosons. From the above formulae we see that the true root multiplicities are given by the coefficients  $c(k\ell - \frac{1}{2}\vec{b}^2)$ .

## 9. An algebra associated to BPS states

The results of sections 4-8 suggest that there is a GKM Lie algebra associated to the BPS states in D=4,N=2 heterotic compactifications. In this section we will construct a GKM Lie super-algebra with even elements associated to BPS vectormultiplets and the odd elements associated to BPS hypermultiplets. For the specific compactifications discussed previously this algebra has root lattice  $\Pi^{1,1}$  or  $\Pi^{9,1}$  and has root multiplicities related to those of the BPS states.

There are two obvious puzzles in trying to construct an algebra of BPS states. First, if we try to define an algebra by looking at the OPE of BPS vertex operators then it is clear that in general the OPE will contain operators for non-BPS states. In particular we will find operators with right-moving conformal dimension  $\bar{h} > 1/2$ . Second, the physical BPS states are connected to a lattice of the form  $\Pi^{s+2,2}$  while the products involved in threshold corrections involve only the lattice  $\Pi^{s+1,1}$ . We will see that the resolution to the first puzzle also provides the resolution to the second.

Consider the first point. In the Neveu-Schwarz sector a vertex operator which creates a BPS state is of the form:

$$V(z, \bar{z}) = e^{ip_L X_L(z) + ip_R \tilde{X}_R(\bar{z})} \Phi_i(z, \bar{z}) \quad (9.1)$$

where  $X_R$  is the bosonic component of the free superfield in the internal  $c = 3$ ,  $N = 2$  theory,  $\Phi$  is an operator of conformal dimensions  $(h, \bar{h} = 1/2)$  and

$$\frac{1}{2}p_L^2 + h - 1 = \frac{1}{2}p_R^2 \quad . \quad (9.2)$$



Note that  $\Phi$  is chiral or anti-chiral primary with respect to the the total right-moving  $N = 2$  SCA with  $U(1)$  charge  $\pm 1$ .<sup>17</sup>

Using the decomposition (3.10) we can assign to each BPS vectormultiplet two distinguished vertex operators whose right-moving chiral primary field in the full internal  $N = 2$  theory is just the unit operator. In the lightcone gauge this vertex operator looks like:

$$V_{vm}^A(z) = e^{ip_L X_L(z) + ip_R \tilde{X}_R(\bar{z})} \Phi^A(z) \tilde{\psi}^\mu(\bar{z}) \quad (9.3)$$

where the index  $A$  runs over an infinite range and  $\tilde{\psi}^\mu(\bar{z})$  is the right-moving transverse spacetime fermion field, i.e.,  $\mu = 2, 3$ . Now in the OPE of two such operators with opposite  $p_R$  we will find terms of the form  $\prod \partial_{\bar{z}}^n X_R$ . These operators create states with right-moving oscillators in the  $c = 3$  theory excited, that is states which are not BPS saturated. At this point we exploit the fact that the  $c = 3$  system is a free system. In (9.3) we can unambiguously drop the right-moving spacetime part of the vertex operator and change the right-moving free fields  $X_R(\bar{z})$  to left-moving free fields which we will denote by  $X_R(z)$  while at the same time retaining the Narain metric on  $(p_L, p_R)$ . Technically, we have defined a mapping from the BPS states in the original light-cone gauge CFT of the heterotic string:<sup>18</sup>

$$\mathcal{C}_{\text{transverse}}^{2,0} \otimes \mathcal{C}_{\text{transverse}}^{0,3} \otimes \mathcal{C}_{\text{free } N=(0,2)}^{0,3} \otimes \mathcal{C}_{N=(0,4)}^{22,6} \quad (9.4)$$

to new leftmoving vertex operators of the form

$$V_{vm}^A(z, \bar{z}) \rightarrow \hat{V}_{vm}^A(z) \equiv e^{ip_L X_L(z) + ip_R X_R(z)} \Phi^A(z) \quad (9.5)$$

which live in the CFT:

$$\mathcal{C}_{\text{transverse}}^{2,0} \otimes \mathcal{C}_{N=(0,4)}^{22,6} \otimes \mathcal{C}_{\text{gaussian}}^{2,0} \quad (9.6)$$

Note that the physical state condition means that the left-moving conformal dimension of these vertex operators is

$$h + \frac{1}{2}p_L^2 - \frac{1}{2}p_R^2 = 1 \quad (9.7)$$

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<sup>17</sup> In the examples we consider explicitly  $X_L$  is either the left-moving partner of  $X_R$  and  $(p_L, p_R) \in \Pi^{2,2}$  or we extend  $X_L$  to include the lattice of the unbroken  $E_8$  and  $(p_L, p_R) \in \Pi^{10,2}$ . In fact our considerations are more general and apply to any theory which has dimension one currents on the left, whether or not they arise from a free field construction, but for simplicity we will restrict ourselves to that case in what follows. We will also ignore cocycle factors in the vertex operators which are necessary to obtain a consistent operator algebra.

<sup>18</sup> Superscripts denote left and right conformal charges.

and hence these operators generate a current algebra based on the lattice  $\Pi^{s+2,2}$ . In terms of CFT correlators the structure constants for three vectormultiplets are

$$\left\langle \hat{V}_{vm}^1(1) \hat{V}_{vm}^2(2) \hat{V}_{vm}^3(3) \right\rangle . \quad (9.8)$$

We have now introduced two negative signature fields into our system and the techniques of BRST cohomology are the most appropriate to use when defining an algebra. We also need a mechanism to kill two sets of oscillator states and at the same time reduce the  $\Pi^{s+2,2}$  lattice to a  $\Pi^{s+1,1}$  lattice. Such a mechanism is provided by string theory with local  $N_{ws} = 2$  world-sheet supersymmetry, particularly in the heterotic version [82]. In heterotic  $N_{ws} = 2$  string theory the leftmoving gauge algebra is a product  $Vir \ltimes U(1)$ . The gauged  $U(1)$  current is of the form  $J = v \cdot \partial X$  with  $v \in \Pi^{s+2,2}$  where  $v$  is null. (For  $\Pi^{2,2}$  and  $\Pi^{10,2}$  there is a unique such null vector up to lattice automorphism.)

In the present context we do not have local  $N_{ws} = 2$  world-sheet supersymmetry but we may nonetheless borrow these ideas to define a Lie algebra based on a  $\Pi^{s+1,1}$  lattice which acts in a positive definite Hilbert space. We let  $L_n$  be the left-moving Virasoro operators constructed from the  $c = (26, 6)$  CFT defined by (9.6) and  $J_n$  the modes of the current defined in the previous paragraph. We then define the physical subspace of states as those annihilated by the  $L_n$  and  $J_n$  for  $n \geq 0$ . The  $J_0$  constraint implies that physical states lie on a  $\Pi^{s+1,1}$  sublattice of the  $\Pi^{s+2,2}$  lattice parametrizing BPS states and the  $L_n$  and  $J_n$  constraints act to remove the negative norm states associated with the two timelike oscillators.

More formally, we enlarge the CFT to the BRST complex

$$\mathcal{C}_{\text{transverse}}^{2,0} \otimes \mathcal{C}_{N=(0,4)}^{22,6} \otimes \mathcal{C}_{\text{gaussian}}^{2,0} \otimes \mathcal{C}_{\xi\eta}^{-2,0} \otimes \mathcal{C}_{b,c}^{-26,0} \otimes \mathcal{C}_x^{2,0} \quad (9.9)$$

where the  $(\xi, \eta)$  system of  $c = -2$  are the  $U(1)$  ghosts. The last factor is a  $c = 2$  pair of free bosonic fields which is needed to ensure we are in the critical dimension. Now we may take the cohomology with respect to  $Q_{vir}$  and  $Q_{U(1)}$  which are defined by:

$$\begin{aligned} Q_{vir} &= \oint \left[ c \left( T_L(z) + \frac{1}{2}(\partial x)^2 - \frac{1}{2}(\partial X_R)^2 + \eta \partial \xi \right) - c \partial c b \right] \\ Q_{U(1)} &= \oint \xi v \cdot \partial X = \sum \xi_n v \cdot \alpha_{-n} \end{aligned} \quad (9.10)$$

where  $T_L(z)$  is the leftmoving stress tensor of the  $\mathcal{C}_{\text{transverse}}^{2,0} \otimes \mathcal{C}_{N=(0,4)}^{22,6}$  system. It is important to work in the “little Hilbert space” of the complex [83]. Thus we should also

take cohomology with respect to  $\oint \eta$ . One important effect of this cohomology is on the allowed momentum. Since

$$\{\oint \eta, \oint \xi v \cdot \partial X\} = \oint v \cdot \partial X \quad , \quad (9.11)$$

momenta must satisfy  $p \cdot v = 0$  and are identified mod  $p \sim p + \lambda v$ . This removes momentum dependence in one  $\Pi^{1,1}$  factor of the lattice. The cohomology with respect to the non-zero modes of  $J$  then kills two of the oscillators as desired. We can now define the vectormultiplet Lie algebra to be the algebra of physical states formed from the cohomology classes of vertex operators  $\hat{V}_{vm}^A(z)$  with respect to the above cohomologies. By the no-ghost theorem the Virasoro cohomology has a positive definite contravariant form and thus the vectormultiplet Lie algebra is a generalized Kac-Moody Lie algebra (or Borchers algebra) [35].

As we saw previously, threshold corrections depend on the difference between vectormultiplets and hypermultiplets. This suggests that we can extend this algebra to a  $Z_2$  graded superalgebra with even elements corresponding to BPS vectormultiplets under the map (9.5) and the odd elements corresponding to hypermultiplets. Threshold corrections would then be interpreted in terms of a supertrace over the states of this superalgebra.

This extension can be accomplished as follows. From (3.10) we see that to each BPS hypermultiplet we can assign two distinguished vertex operators whose right-moving chiral primary field is a highest weight state for the  $\mathcal{A}_{N=4}^{c=6}$  algebra and the unit operator for the  $\mathcal{A}_{N=2}^{c=3}$  algebra.

$$V_{hm}^{A,i}(z, \bar{z}) = e^{ip_L X_L(z) + ip_R \tilde{X}_R(\bar{z})} \Phi^{A,i}(z, \bar{z}) \quad (9.12)$$

where  $A$  runs over an infinite range and  $i$  over a finite range (labelling the degeneracy of the  $(\frac{1}{2}, \frac{1}{2})$  representation of the  $N = 4, c = 6$  theory.) We again construct a new vertex operator by making  $X_R$  left-moving and dropping the unit operator in the  $c = 3$  theory:

$$V_{hm}^{A,i}(z, \bar{z}) \rightarrow \hat{V}_{hm}^{A,i}(z, \bar{z}) \equiv e^{ip_L X_L(z) + ip_R X_R(z)} \Phi^{A,i}(z, \bar{z}) \quad (9.13)$$

This gives a vertex operator which is a holomorphic vertex operator multiplied by an anti-holomorphic operator which is a primary chiral operator in the  $\mathcal{A}_{N=4}^{c=6}$  algebra. To define a closed algebra we need to combine the left-moving current algebra with the graded commutative associative chiral algebra on the right. Since the  $N = 2$  chiral algebra splits according to (2.1), the bosonized  $U(1)$  current also splits:

$$\sqrt{3}H^{tot} = H + \sqrt{2}H^{N=4} \quad (9.14)$$

so that the  $N = 4$  current is bosonized by: <sup>19</sup>

$$-2J_{+-} = -i\sqrt{2}\partial H^{N=4} \quad (9.15)$$

Since the  $U(1)$  charge violation of the  $N = 4$  system is 2 the chiral ring of the  $N = 4$  theory gives us a map from  $\hat{V}_{hm}^{A,i} \times \hat{V}_{hm}^{B,j} \rightarrow V_{vm}^C$ , thus giving a  $\mathbb{Z}_2$  grading of the desired type. Explicitly, the structure constants between two hypermultiplets and one vectormultiplet are:

$$\left\langle \hat{V}_{hm}^1(1) \hat{V}_{hm}^2(2) \hat{V}_{vm}^3(3) e^{-i\sqrt{2}H^{N=4}}(\bar{z}_4) \xi(z_0) \right\rangle \quad (9.16)$$

where  $z_0, z_4$  can be inserted at any point  $\neq z_1, z_2, z_3$ .

More formally, we can take a twisted  $N = 4$  cohomology [84,85,86] on the right in addition to the Virasoro and  $U(1)$  cohomologies on the left. From the (small)  $N = 4$  superconformal algebra, we have four odd supercurrents  $G_{AB}$ ,  $A, B = +, -$ . Now we consider a twisting so that  $T' = T - \partial J_{+-}$ . Then  $G_{+A}$  has  $h' = 1$  and  $G_{-A}$  has  $h' = 2$ . Thus, there are two BRST currents. In defining the topological states we should take *both* cohomologies

$$\tilde{Q}_A = \oint G_{+A} \quad (9.17)$$

The resulting cohomology states are chiral primary (not antichiral primary) with respect to all embedded  $N=2$  algebras. On the resulting cohomology we can combine the leftmoving “Gerstenhaber bracket” with the rightmoving “Gerstenhaber product,” [87][88], that is, we can take the product:

$$(V^1, V^2)(w, \bar{w}) \rightarrow \oint_w dz \lim_{\bar{z} \rightarrow \bar{w}} b_{-1} V^1(z, \bar{z}) \cdot V^2(w, \bar{w}) \quad (9.18)$$

Under this product the right-moving piece of two hypermultiplets maps to a right-moving chiral primary with charge +2. This does not correspond to a BPS vertex operator, but we can conjugate, or, equivalently, use the topological metric as in (9.16) above.

The vertex operators  $\hat{V}$  associated to BPS states then form a graded Lie algebra of the generalized Kac-Moody type. If we introduce vectormultiplets with a multiplicity two then the root supermultiplicities of this algebra coincide with the degeneracies  $c(-r^2/2)$  occurring in the product formula.

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<sup>19</sup> In terms of the  $N = 2$   $U(1)$  current,  $J^{(2)} = 2J_{+-}$ .

## 10. Application to $N = 2$ String Duality

There is now some evidence that  $N = 2$  heterotic string theories are dual to Type IIA or IIB string theory on special Calabi-Yau spaces [9,10]. These Calabi-Yau spaces have, roughly speaking, the structure of  $K3$  surfaces fibered over rational curves [89][57]. The results presented here should allow for detailed tests of this duality in a broader class of models than has been considered so far. We hope to return to this issue in more detail elsewhere [90] but will make a few preliminary remarks here.

The heterotic theory we have considered arises from the symmetric embedding of the spin connection in the gauge group. In the  $Z_2$  orbifold limit that we have used for explicit computations the massless spectrum consists of vector multiplets transforming in the adjoint representation of  $E_8 \times E_7 \times SU(2) \times U(1)^4$  with a total of 388 states (including the graviphoton) and hypermultiplets transforming as

$$4(1, 1, 1) + 8(1, 56, 1) + (1, 56, 2) + 32(1, 1, 2). \quad (10.1)$$

with a total of 628 states. Note that  $388 - 628 = -240$  which matches the coefficient  $c_1(0)$  in the  $s = 0$  case with unbroken  $E_8$  gauge group. In order to compare with the spectrum of a possible dual Calabi-Yau space we can completely Higgs the  $E_7 \times SU(2)$  gauge group (which does not change the difference between the number of vector and hypermultiplets) and break  $E_8$  to  $U(1)^8$  by turning on Wilson lines. This leaves us with gauge group  $U(1)^{12}$  and 492 gauge neutral hypermultiplet fields [9,91]. The difference between the number of massless vector and hypermultiplets is then  $12 - 492 = -480$  which agrees with the coefficient  $c_1(0)$  for  $s = 8$ .

This massless spectrum would arise in Type II string theory on a Calabi-Yau space with  $b_{11} = 11$  and  $b_{21} = 491$  and hence with Euler number  $\chi = -960$ . Precisely such a Calabi-Yau family, denoted  $X_{84}^{1,1,12,28,42}$  appears in the list of  $K3$  fibrations given in [57] and is distinguished by having the maximal value of  $|\chi|$  known for Calabi-Yau spaces.<sup>20</sup> These Calabi-Yau's are resolutions of degree 84 hypersurfaces in  $\mathbb{P}_4^{1,1,12,28,42}$ . A typical defining polynomial would be:

$$x_1^{84} + x_2^{84} + x_3^7 + x_4^3 + x_5^2 = 0. \quad (10.2)$$

Following the procedure in [89] we obtain a  $K3$  fibration in the sense that there is a complete linear system  $|L|$  whose divisors are  $K3$  surfaces. In the present example we set

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<sup>20</sup> We are grateful to D. Morrison for some very helpful remarks concerning these spaces.

$x_2 = \lambda x_1$  and define  $y_1 = x_1^2$  to get a family  $X_{42}^{1,6,14,21}$  of K3 surfaces realized as degree 42 polynomials in  $\mathbb{P}_3^{1,6,14,21}$ . For example (10.2) gives:

$$(1 + \lambda^{84})y_1^{42} + x_3^7 + x_4^3 + x_5^2 = 0 \quad (10.3)$$

The K3 family  $X_{42}^{1,6,14,21}$  possesses many beautiful and special properties, and has arisen before in the physics literature [92][93]. According to [93] the family is self-mirror with complexified Kahler cone  $\cong \mathcal{H}^{9,1}$ .

The formulae we have derived for the heterotic prepotential fit in well with the predictions of heterotic/type II duality. According to [9][57] the cohomology class in  $H^2(X; \mathbb{Z})$  dual to a divisor  $L$  in the linear system defines the heterotic coordinate  $S$ . The remaining generators of the Kähler cone define coordinates  $y \in \mathcal{H}^{9,1}$  and may be identified with the Kähler classes of  $X_{42}^{1,6,14,21}$ . The form of the  $S$ -dependence,  $Sy^2$  of the classical prepotential follows from the structure of the K3-fibration, therefore, let us consider the third derivatives with respect to the  $y^A$ . The third derivative of the perturbative prepotential following from (4.25) is

$$\frac{\partial}{\partial y^A} \frac{\partial}{\partial y^B} \frac{\partial}{\partial y^C} \mathcal{F}^{\text{pert}} = \frac{\tilde{d}_{ABC}^{10,2}}{64\pi^2} + \frac{1}{2\pi} \sum_{r>0} c_1(-r^2/2) r_A r_B r_C \frac{e^{-2\pi r \cdot y}}{1 - e^{-2\pi r \cdot y}} \quad (10.4)$$

Note this is in precisely the right form expected for the counting of rational curves on a Calabi-Yau. Substituting the values of  $\tilde{d}_{ABC}^{10,2}$  from the appendix we see that the third derivatives of  $\mathcal{F}$  are indeed integers if we multiply  $\mathcal{F}$  by  $2\pi t$  where  $t$  is an integer.<sup>21</sup> Moreover, since  $c_1(0) = \frac{1}{2}\chi$  we now recognize the constant term in (4.25):

$$-\frac{2\delta}{384\pi^2} = -\frac{c_1(0)\zeta(3)}{2(2\pi)^4}, \quad (10.5)$$

as the famous  $\zeta(3)$  term appearing in the Calabi-Yau prepotential of [94] which is related to the four-loop renormalization of the Calabi-Yau sigma-model [95]. Comparing with the normalization of [94] we see that the properly normalized prepotential must be  $4\pi\mathcal{F}$ .

In view of this heterotic/type II duality makes some curious predictions for algebraic geometry, namely:

The cubic intersection product on  $H^{1,1}$  for the family  $X_{84}^{1,1,12,28,42}$  is governed by the structure of  $E_8$  and may be computed from the expressions in (A.51) and (A.52)

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<sup>21</sup> In verifying this it is important to bear in mind the freedom to redefine  $\mathcal{F}$  by (4.34). We hope to further investigate this chamber-dependence in the future [90].

by differentiation. Moreover, the rational curves in the family  $X_{84}^{1,1,12,28,42}$  which are orthogonal to  $c_1(L) \in H^2(X; \mathbb{Z})$  are parametrized by the positive roots  $r > 0$  of  $E_{10}$  and appear with multiplicity  $2c_1(-r^2/2)$  where  $c_1(n)$  are the coefficients of the modular form  $F(q) = E_6/\eta^{24}$ .

If the rational curves are not isolated the integers must be interpreted as integrals over the moduli space, as is standard in topological field theory. These integers should be related to the numbers of rational curves on  $K3$  surfaces in  $X_{42}^{1,6,14,21}$ .

Remarks entirely analogous to the above apply to the relation between the family of  $K3$  fibrations  $X_{24}^{1,1,2,8,12}$  discussed in [9][57] and the formula (4.36).

## 11. Concluding Speculations

We have shown that threshold corrections in  $N = 2$  heterotic string theories are determined by the spectrum of BPS states and have provided evidence that there is a generalized Kac-Moody algebra associated with these states which governs the form of the threshold corrections. Various extensions of these results should prove very interesting.

It would be interesting to generalize the computations here to other backgrounds and more general dependence on the moduli. In particular, the computations done here involved the standard embedding of the spin connection on  $K3$  in the gauge group. By repeating these calculations for different ranks of gauge group and for different topologies of gauge bundle it should be possible to recover interesting product formulae associated with a large number of hyperbolic and generalized Kac-Moody algebras. As examples, we expect to get an interesting algebra associated with  $\Pi^{17,1}$  even with the symmetric embedding, and moreover we expect to recover the products occurring in the work of Feingold and Frenkel [96]. Indeed, the relevant products have already been suggested in [59].

Other embeddings and other choices of moduli dependence will also lead to modular integrals for congruence subgroups of  $SL(2, \mathbb{Z})$ . Given the presence (for the standard embedding) of the Monster Lie algebra and the  $j$  function, which is the Thompson series for the identity element of the Monster, it seems likely that these other embeddings will involve some of the other Thompson series for the Monster. It might be that the Monster provides a general classification of  $N = 2$  heterotic string vacua.

The formula we have given for the perturbative heterotic prepotential fits in well with the proposed heterotic/type II string duality and therefore admits a natural extension to a

full nonperturbative answer. In this paper we have shown that the special Kahler geometry of the  $N = 2$  heterotic compactification is summarized by the prepotential:

$$\begin{aligned} \mathcal{F} = & -S(y, y) + \frac{1}{384\pi^2} i \tilde{d}_{ABC}^{s+2,2} y^A y^B y^C - \frac{\zeta(3)}{2(2\pi)^4} c_1(0) \\ & - \frac{1}{(2\pi)^4} \sum_{r>0} c_1(-r^2/2) Li_3(e^{2\pi i r \cdot y}) + \mathcal{F}^{\text{nonpert}}(y, e^{2\pi i S}) \end{aligned} \quad (11.1)$$

where  $\mathcal{F}^{\text{nonpert}}(y, e^{2\pi i S})$  has an analytic power series expansion in its second argument. As noted in the previous section, the formula for the prepotential as a sum of trilogarithms is *dictated* by the curve-counting formulae given a dual type II background. Thus, for backgrounds admitting a dual pair, the full nonperturbative prepotential will again be a sum of trilogarithms, with the replacement

$$e^{2\pi i r \cdot y} \rightarrow e^{2\pi i r \cdot y + 2\pi i n S}$$

in the argument of the trilogarithms. The sum will run over the full positive Kahler cone of the dual Calabi-Yau variety. In view of our results it is natural to speculate that this sum will again be a sum over positive roots of some interesting algebra.

The results of this paper should have two interesting mathematical applications in the context of heterotic/type II string duality. First, given an  $N = 2$  dual string pair the results presented here combined with those of [97] suggest that special combinations of Ray-Singer torsions on dual Calabi-Yau spaces should admit infinite product representations. These considerations are undoubtedly related to the recent work of Jorgenson and Todorov [98]. Second, interesting recent work of Lian and Yau [99], has shown that certain mirror maps associated to one-parameter families of algebraic K3 surfaces are related to Thompson series. Since the heterotic string is a more natural home for the Monster it is tempting to speculate that the algebraic structure of BPS states together with string duality might provide a way to understand the observations of [99].

The algebraic structures we have begun to uncover should also shed new light on string duality. For example, it is tempting to speculate that different dual theories are simply different representations of an underlying algebraic structure much as in the different realizations of affine Kac-Moody algebras. It would be very interesting to try to identify generalized Kac-Moody algebras in Type II string theories on Calabi-Yau spaces arising as  $K3$  fibrations. In addition to the applications to  $N = 2$  string duality outlined in the previous section our work might also have relations to the conjectured duality between



heterotic and Type I theories [14]. The exchange of  $\bar{z}$  and  $z$  dependence needed to define the vertex algebra is reminiscent of world-sheet orbifolds considered in [100,101,102] and recently applied to Type I -heterotic duality in [103]. There is possibly an alternative point of view on our construction which might be interesting to pursue. In closed string theory there is a Virasoro algebra which generates transformations on the spatial string coordinate  $\sigma$  with generators  $\hat{L}_n = L_n - \bar{L}_{-n}$ . If we take the  $\bar{L}_{-n}$  to be those of the free right-moving bosons and the  $L_n$  those of the left-moving degrees of freedom then the vertex operators we have defined are dimension one as a result of the condition (9.7). If we were then to complexify  $\sigma$  we would obtain a holomorphic vertex operator algebra. This suggests a complexification of the string world-sheet, something that has also been suggested in other contexts [104].

Finally, if the GKM algebra associated to BPS states that we have found is a gauge algebra, as it appears, then in analogy to the structure of  $N = 2$  Yang-Mills theory the full nonperturbative prepotential should be governed by “monopoles” of this algebra and the monodromies given by an algebraic variety whose monodromy group is the Weyl group of this algebra. There is a natural candidate for these monopoles. Recall that the gauge structure became clearest in the  $M_{\text{string}} \rightarrow 0$  limit, which coincides with a decompactification to 5 dimensions. We propose that the states being counted by the product representation of threshold corrections are five-branes wrapped around the remaining  $K3 \times S^1$  internal space.

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### Appendix A. Fundamental Domain Integrals

In this appendix we evaluate the following two integrals:

$$\mathcal{I}_{s+2,2}(y) \equiv \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \left[ \left( \sum_{p \in \Gamma^{s+2,2}} q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}p_R^2} \right) F(q) - c(0) \right] \quad (\text{A.1})$$

and

$$\tilde{\mathcal{I}}_{s+2,2}(y) \equiv \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \left[ \left( \sum_{p \in \Gamma^{s+2,2}} q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}p_R^2} \right) F(q) \left( E_2 - \frac{3}{\pi\tau_2} \right) - \tilde{c}(0) \right] \quad (\text{A.2})$$

These integrals are absolutely convergent for  $y \in \mathcal{H}^{s+1,1}$  where  $y$  is not on the RQD's. They have logarithmic singularities on the RQD's and moreover, are  $O(s+2, 2; \mathbb{Z})$  invariant.

The notation is as follows:  $\mathcal{F}$  is the fundamental domain for  $SL(2, \mathbb{Z})$ .  $F(q)$  is a  $SL(2, \mathbb{Z})$  modular function of weight  $-s/2$  for  $\mathcal{I}$  or of weight  $-s/2 - 2$  for  $\tilde{\mathcal{I}}$  which is holomorphic except at infinity where it has a first order pole. That is, we assume  $F$  has a Fourier expansion in terms of  $q = e^{2\pi i\tau}$ :

$$F(q) = \sum_{n=-1}^{\infty} c(n)q^n = c(-1)q^{-1} + c(0) + \dots \quad (\text{A.3})$$

and define  $c(n) \equiv 0$ ,  $n < -1$ . We also define coefficients  $\tilde{c}(n)$  through the expansion:

$$F(q)E_2(q) \equiv \sum_{n=-1}^{\infty} \tilde{c}(n)q^n \quad (\text{A.4})$$

where  $E_2(q)$  is the first of the series of Eisenstein functions:

$$E_2(q) = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)q^n, \quad (\text{A.5})$$

$$E_4(q) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n} = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n, \quad (\text{A.6})$$

$$E_6(q) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^n} = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^n, \quad (\text{A.7})$$

where

$$\sigma_k(n) = \sum_{d|n} d^k \quad (\text{A.8})$$

is the sum of the  $k$ th powers of the divisors of  $n$ .

$E_2(\tau)$  is not modular covariant since it transforms with a shift. The combination

$$E_2(\tau) - \frac{3}{\pi\tau_2} \quad (\text{A.9})$$

appearing in (A.2) on the other hand is not holomorphic but transforms covariantly.

The sums in (A.1) and (A.2) run over an even self-dual lattice  $\Gamma^{s+2,2}(y) \cong \Pi^{s+2,2}$ , and thus  $s = 8t$  with  $t$  integer. As discussed in the text we write  $\Pi^{8t+2,2} \cong \Pi^{8t,0} \oplus \Pi^{2,2}$  and write lattice vectors as

$$(\vec{b}; m_+, n_-; m_0, n_0) \quad (\text{A.10})$$

with metric

$$(\vec{b}; m_+, n_-; m_0, n_0)^2 = \vec{b}^2 - 2m_+n_- + 2m_0n_0 \quad (\text{A.11})$$

Also, for Narain compactifications we have

$$\begin{aligned} \frac{1}{2}(\vec{p}_L^2 - \vec{p}_R^2) &= \frac{1}{2}\vec{b}^2 - m_+n_- + m_0n_0 \\ \frac{1}{2}p_R^2 &= \frac{1}{-2(\Im y)^2} \left| \vec{b} \cdot \vec{y} - m_+y_- - n_-y_+ + m_0 - \frac{1}{2}n_0y^2 \right|^2 \end{aligned} \quad (\text{A.12})$$

Our conventions for the complex coordinates on the homogeneous space  $SO(8t+2, 2)/SO(8t+2) \times SO(2)$  are as in sec. 2.

#### A.1. Evaluation of $\mathcal{I}_{s+2,2}$

We will follow quite closely the calculation in the appendix of [42]. The general strategy for the evaluation of (A.1) and (A.2) is to perform first a Poisson resummation on the “momenta”  $m_+$  and  $m_0$  which leads to a sum over matrices  $A$  which are general two by two matrices with integer elements. The contributions of two matrices  $A, A'$  related by an element  $V$  of  $SL(2, \mathbb{Z})$  are related by modular transformation of  $\tau$  by  $V$  which allows one to sum instead over orbits of  $SL(2, \mathbb{Z})$  and integrate over the images of the fundamental domain under the elements  $V$  that yield distinct matrices  $A$  when acting on a representative element of the orbit.

We now consider (A.1). Performing a Poisson resummation on  $m_+, m_0$  gives

$$\tau_2 Z_\Gamma = \tau_2 \sum_{p \in \Gamma^{s+2,2}} q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}p_R^2} = \sum_{A \in Mat_{2 \times 2}} T_\Gamma[A] \quad (\text{A.13})$$

where

$$T_\Gamma[A] = \frac{-(\Im y)^2}{2y_{-,2}} \sum_{\vec{b} \in \Gamma^{8s,0}} q^{\frac{1}{2}\vec{b}^2} \exp \mathcal{G}, \quad (\text{A.14})$$

$$\begin{aligned} \mathcal{G} &= \left[ \frac{\pi(\Im y)^2}{2y_{-,2}^2 \tau_2} |\mathcal{A}|^2 - 2\pi i y_+ \det A + \frac{\pi}{y_{-,2}} \vec{b} \cdot (\vec{y} \tilde{\mathcal{A}} - \vec{y}^* \mathcal{A}) \right. \\ &\quad \left. - \frac{\pi}{2y_{-,2}} n_0 (\vec{y}^2 \tilde{\mathcal{A}} - (\vec{y}^*)^2 \mathcal{A}) + \frac{i\pi}{y_{-,2}^2} (\Im \vec{y})^2 (n_- + n_0 y_-^*) \mathcal{A} \right] \end{aligned} \quad (\text{A.15})$$

and

$$\begin{aligned}\mathcal{A} &= \begin{pmatrix} 1 & y_- \end{pmatrix} \begin{pmatrix} n_- & k_1 \\ n_0 & -k_2 \end{pmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & y_- \end{pmatrix} A \begin{pmatrix} \tau \\ 1 \end{pmatrix} \\ \tilde{\mathcal{A}} &= \begin{pmatrix} 1 & y_-^* \end{pmatrix} \begin{pmatrix} n_- & k_1 \\ n_0 & -k_2 \end{pmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix}\end{aligned}\tag{A.16}$$

Under modular transformations we have

$$\begin{aligned}A &\rightarrow A \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ \mathcal{A} &\rightarrow \frac{1}{c\tau + d} \mathcal{A}\end{aligned}\tag{A.17}$$

Unlike the simpler integral analyzed in [42], it is no longer obvious that the contribution from two matrices  $A$  related by the modular transformation (A.17) is given by the modular transformation  $\tau' = \frac{a\tau+b}{c\tau+d}$ . However for  $s = 8t$  it is still true as can be seen by Poisson resummation on the  $\Pi^{8t,0}$  lattice sum and using the fact that  $\Pi^{8t,0}$  is even self-dual. Note that one could equally well have used a Poisson resummation on  $n_-, m_0$  leading to the same expression with  $y_+$  exchanged for  $y_-$ .

Following [42] we can split the sum on  $A$  into three orbits and correspondingly write the integrals as a sum of three integrals:

$$\begin{aligned}\mathcal{I}_{s+2,2} &= \mathcal{I}_{s+2,2}^0 + \mathcal{I}_{s+2,2}^{nd} + \mathcal{I}_{s+2,2}^{dg} \\ \mathcal{I}_{s+2,2}^0 &= \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} T_{\Pi}[A=0] F(q) \\ \mathcal{I}_{s+2,2}^{nd} &= 2 \int_{\mathcal{H}} \frac{d^2\tau}{\tau_2^2} \sum_{0 \leq j < k, p \neq 0} T_{\Pi}[A = \begin{pmatrix} k & j \\ 0 & p \end{pmatrix}] F(q) \\ \mathcal{I}_{s+2,2}^{dg} &= \int_{\mathcal{S}} \frac{d^2\tau}{\tau_2^2} \left[ \sum'_{j,p} T_{\Pi}[A = \begin{pmatrix} 0 & j \\ 0 & p \end{pmatrix}] F(q) - \tau_2 c(0) \theta(\tau \in \mathcal{F}) \right]\end{aligned}\tag{A.18}$$

where  $\mathcal{H}$  is the upper half plane and  $\mathcal{S}$  is the strip  $\{\tau \in \mathcal{H}, |\tau_1| < 1/2\}$ . These three terms correspond to the zero orbit  $A = 0$ , the non-degenerate orbit with  $\det A \neq 0$  and the degenerate orbit with  $\det A = 0$  and a particular choice of representative matrices from each orbit.

In deriving (A.18) we have made an important exchange of summation and integration. Because of the tachyon divergences  $\sim q^{-1}$  which exist before the  $L_0 = \bar{L}_0$  projection this exchange can be invalid. Indeed, it is clear that if  $y_{+,2} > 2y_{-,2}$  then one must use the expressions (A.13),(A.14)(A.15). On the other hand, if  $y_{-,2} > 2y_{+,2}$ , one must use the

other Poisson summation with  $y_+ \leftrightarrow y_-$ . In fact, we know that the expressions must change as we cross the wall of the Weyl chamber  $y_{-,2} = y_{+,2}$ .<sup>22</sup>

The  $A = 0$  orbit may be evaluated using [105]<sup>23</sup>.

$$\frac{-(\Im y)^2}{2y_{-,2}} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \vartheta_{\Pi^{st,0}} F(q) = \frac{-(\Im y)^2}{2y_{-,2}} \frac{1}{\pi} [G_2(q) \vartheta_{\Pi^{st,0}} F(q)]_{q^0} \quad (\text{A.19})$$

For  $t = 1$  which will be of most relevance for our discussion we have

$$\frac{1}{\pi} [G_2(q) \vartheta_{\Pi^{s,0}} F(q)]_{q^0} = \frac{\pi}{3} (c(0) + 216c(-1)) \quad (\text{A.20})$$

The non-degenerate orbit is evaluated using the representative matrix

$$A_0 = \begin{pmatrix} k & j \\ 0 & p \end{pmatrix} \quad (\text{A.21})$$

with  $0 \leq j < k$  and  $p \neq 0$ . The  $\tau_1$  integral is gaussian and the sum on  $j$  is then trivial. The integral over  $\tau_2$  gives a representation of the Bessel function  $K_{1/2}$  which is elementary. The sum on  $p$  yields a logarithm so that the contribution from the non-degenerate orbit is given by

$$-2 \log \left| \prod_{\vec{b} \in \Pi^{st,0}} \prod_{k>0, \ell \in \mathbb{Z}} \left( 1 - e^{2\pi i r \cdot \hat{y}} \right)^{c(k\ell - \frac{1}{2} \vec{b}^2)} \right|^2 \quad (\text{A.22})$$

where we have introduced a “hatted dot product” defined by

$$r \cdot \hat{y} \equiv \Re \left[ (\vec{b} \cdot \vec{y} + \ell y_- + k y_+) \right] + i \left| \Im \left[ (\vec{b} \cdot \vec{y} + \ell y_- + k y_+) \right] \right| \quad (\text{A.23})$$

when  $k > 0$ , and, when  $k = 0$ :

$$\begin{aligned} r \cdot \hat{y} &\equiv r \cdot y - N(r, y) y_- \\ N(r, y) &= \text{sgn}(\vec{b}) \left[ \text{sgn}(\vec{b}) \frac{\vec{b} \cdot \Im \vec{y}}{y_{-,2}} \right] \end{aligned} \quad (\text{A.24})$$

where  $[\cdot]$  is the greatest integer function.

To evaluate the contribution from the degenerate orbit we take

$$A_0 = \begin{pmatrix} 0 & j \\ 0 & p \end{pmatrix} \quad (\text{A.25})$$

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<sup>22</sup> Hence there is a puzzle about why the exchange of sum and integral is not valid for one Poisson summation in the range  $2 > y_{+,2}/y_{-,2} > 1$ . We have not resolved this point.

<sup>23</sup> After correcting a factor two error in eqn. 9.38 of [105].

with  $j, p$  not both zero. After evaluating the  $\tau_1$  integral the sum on  $j$  may be performed using a Sommerfeld-Watson transformation:

$$\begin{aligned} \sum_{j=-\infty}^{\infty} \frac{e^{i\theta j}}{(j+B)^2 + C^2} &= \frac{\pi}{C} e^{-i\theta(B-iC)} \frac{1}{1 - e^{-2\pi i(B-iC)}} \\ &\quad + \frac{\pi}{C} e^{-i\theta(B+iC)} \frac{e^{2\pi i(B+iC)}}{1 - e^{2\pi i(B+iC)}} \\ C &> 0, \quad 0 \leq \theta \leq 2\pi \end{aligned} \quad (\text{A.26})$$

A special case is

$$\sum_{j=1}^{\infty} \frac{\cos \theta j}{j^2} = \frac{\theta(\theta - 2\pi)}{4} + \frac{\pi^2}{6} \quad (\text{A.27})$$

for  $0 \leq \theta \leq 2\pi$ .

To write the final answer it is useful to introduce the polylogarithm functions:

$$\begin{aligned} Li_1(x) &= \sum_{j=1}^{\infty} \frac{x^j}{j} = -\log(1-x) \\ Li_2(x) &= \sum_{j=1}^{\infty} \frac{x^j}{j^2} = \int_0^1 \frac{dt}{t} \frac{1}{(1-xt)} \\ Li_3(x) &= \sum_{j=1}^{\infty} \frac{x^j}{j^3} = -\int_0^1 \frac{dt}{t} \int_0^1 \frac{ds}{s} \log(1-xts) \end{aligned} \quad (\text{A.28})$$

The general answer for the integral (A.1) is then

$$\begin{aligned} \mathcal{I}_{8t+2,2}(y) &= \frac{-(\Im y)^2}{2y_{-,2}} \frac{\pi}{3} [E_2(q) \vartheta_{\Pi^{st,0}} F(q)]_{q^0} \\ &\quad + c(0) \left( -\log[-(\Im y)^2] + \frac{\pi}{3} y_{-,2} - \mathcal{K} \right) \\ &\quad + c(-1) \frac{2y_{-,2}}{\pi} \sum_{\vec{b}^2=2} \Re \left[ Li_2 \left( e^{2\pi i \frac{\vec{b} \cdot \Im \vec{y}}{y_{-,2}}} \right) \right] \\ &\quad + 4\Re \left\{ \sum_{r>0} \left[ c(kl - \frac{1}{2} \vec{b}^2) Li_1(e^{2\pi i r \cdot \hat{y}}) \right] \right\} \end{aligned} \quad (\text{A.29})$$

Here  $r > 0$  means:

1.  $k > 0$  or,
2.  $k = 0, \ell > 0$  or,

3.  $k = \ell = 0, \vec{b} > 0$ .

The first line of (A.29) comes from  $A = 0$ . The second and third lines come from the degenerate orbit, and the last line comes from the nondegenerate and degenerate orbits. The constant  $\mathcal{K}$  is

$$\mathcal{K} = \log\left[\frac{4\pi}{\sqrt{27}}e^{1-\gamma_E}\right] \quad (\text{A.30})$$

where  $\gamma_E$  is the Euler-Mascheroni constant.

We now consider two special cases.

$s = 0$ : In this case it is customary to denote  $y_+ = T, y_- = U$ . The general formula (A.29) then gives

$$\begin{aligned} \mathcal{I}_{2,2}(T, U) = & c(0)[- \log[2T_2U_2] - \mathcal{K}] + c(0)\frac{\pi}{3}(T_2 + U_2) \\ & - 2 \log \left| e^{-2\pi i T c(-1)} \prod_{r>0} \left( 1 - e^{2\pi i (kT + \ell U)} \right)^{c(k\ell)} \right|^2 \end{aligned} \quad (\text{A.31})$$

Here  $r > 0$  means  $k > 0, \ell \in \mathbb{Z}$ , or  $k = 0, \ell > 0$ . We have also assumed that  $T_2 > U_2$ . In this case one may replace  $r \hat{\cdot} y$  with  $r \cdot y$  since they only differ for  $k = 1, \ell = -1$  and in this case  $r \hat{\cdot} y = (T_1 - U_1) + i|T_2 - U_2|$  which equals  $r \cdot y$  for  $T_2 > U_2$ . For  $U_2 > T_2$  one should interchange  $U$  and  $T$  in (A.31).

The last two terms in (A.31) may be written as

$$-2 \log |j(T) - j(U)|^2 - 2 \log |\eta(T)\eta(U)|^{2c(0)} \quad (\text{A.32})$$

using the product formula (1.2).

$s = 8$ : In this case we have, without loss of generality,

$$F(q) = \frac{E_4^2}{\eta^{24}} \quad (\text{A.33})$$

The finite sums over simple roots of  $\Pi^{8,0}$  can be simplified. In (A.29) we have a term

$$\Re \left[ Li_2(e^{i\theta}) \right] = \sum_{j=1}^{\infty} \frac{\cos \theta j}{j^2} = \frac{\theta(\theta - 2\pi)}{4} + \frac{\pi^2}{6} \quad 0 \leq \theta \leq 2\pi \quad (\text{A.34})$$

In order to apply this to the third line of (A.29) we need to be careful to take care of the range of the angle  $\frac{\vec{b} \cdot \Im \vec{y}}{y_{-,2}}$ . Since, by assumption,  $y$  is not at an enhanced symmetry point this quantity is not integral. If we further take  $y$  to be in the fundamental Weyl chamber (5.12) we get:

$$2c(-1) \frac{2y_{-,2}}{\pi} \sum_{\vec{b}^2=2, \vec{b}>0} \left[ \frac{\pi^2}{3} - 2\pi^2 \frac{\vec{b} \cdot \Im \vec{y}}{y_{-,2}} + 2\pi^2 \left( \frac{\vec{b} \cdot \Im \vec{y}}{y_{-,2}} \right)^2 \right] \quad (\text{A.35})$$

We can evaluate the sum over  $E_8$  root vectors. The Weyl vector of  $E_8$  is  $\vec{\rho} \equiv \frac{1}{2} \sum_{\vec{b}^2=2, \vec{b}>0} \vec{b}$  and the quadratic sum can be evaluated [106] to give

$$\sum_{\vec{b}^2=2} (\vec{b} \cdot \vec{v})^2 = 60\vec{v}^2 \quad (\text{A.36})$$

To summarize, under the assumption (5.12) we get the product:

$$\begin{aligned} \mathcal{I}_{10,2}(y) &= -2 \log |\Phi(y)|^2 + c(0) \left( -\log[-(\Im y)^2] - \mathcal{K} \right) \\ \Phi(y) &= e^{2\pi i \rho \cdot y} \prod_{r>0} \left( 1 - e^{2\pi i r \cdot y} \right)^{c(k\ell - \frac{1}{2}\vec{b}^2)} \\ \rho &= -(\vec{\rho}; 31, 30) \end{aligned} \quad (\text{A.37})$$

In general the integral  $\mathcal{I}_{s+2,2}(y)$  has a Weyl vector. This means that the cubic terms cancel, as can be derived using theorems 6.2 and 10.3 of Borchers [29]. Moreover, the Weyl vector  $\rho$  appearing in the product formulae of [29] can be extracted from the terms linear in  $y$ . Abstractly we have - for all  $t \geq 1$ :

$$y_{+,2} \frac{\pi}{3} [E_2 \vartheta_{\Pi^{st,0}} F(\tau)]_{q^0} + y_{-,2} \frac{\pi}{3} [\vartheta_{\Pi^{st,0}} F(\tau)]_{q^0} - 4\pi \sum_{\vec{b}>0} c(-\vec{b}^2/2) \vec{b} \cdot \Im \vec{y}. \quad (\text{A.38})$$

This agrees exactly with the expression in theorem 10.4 of [29].

## A.2. Evaluation of $\tilde{\mathcal{I}}$

We now turn to an evaluation of (A.2). We again decompose the sum over  $A$  into a sum over orbits of  $SL(2, \mathbb{Z})$ . The  $A = 0$  orbit is done using [105]

$$\int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2^2} (\hat{G}_2(\tau))^k F(q) = \frac{1}{\pi(k+1)} \left[ (G_2(\tau))^{k+1} F(q) \right]_{q^0} \quad (\text{A.39})$$

The evaluation of the non-degenerate orbit proceeds as before except that the  $\tau_2$  integral now gives a representation of  $K_{3/2}$  and in the sum over  $p$  one makes the replacement

$$\frac{1}{|p|} \rightarrow \frac{\left| \Im \left[ (\vec{b} \cdot \vec{y} + \ell y_- + k y_+) \right] \right|}{p^2} + \frac{1}{2\pi |p|^3} \quad (\text{A.40})$$



Thus, instead of logarithms we get  $Li_2(x)$  and  $Li_3(x)$ . The evaluation of the degenerate orbit also proceeds as before except that the sum on  $j$  now involves

$$\sum_{j=-\infty}^{\infty} \frac{e^{i\theta j}}{((j+B)^2 + C^2)^2} = -\frac{1}{2C} \frac{\partial}{\partial C} \sum_{j=-\infty}^{\infty} \frac{e^{i\theta j}}{((j+B)^2 + C^2)} \quad (\text{A.41})$$

which thus reduces to (A.26). The result is

$$\begin{aligned} \tilde{\mathcal{I}}_{8t+2,2}(y) &= \frac{-(\Im y)^2}{2y_{-,2}} \frac{\pi}{6} [E_2^2(q) \vartheta_{\Pi^{8t,0}} F(q)]_{q^0} + \tilde{c}(0) (-\log[-(\Im y)^2] + \frac{\pi}{3} y_{-,2} - \mathcal{K}) \\ &+ c(0) \frac{2\pi}{15} \frac{y_{-,2}^3}{(\Im y)^2} + \frac{6}{\pi^2} \frac{c(0)}{(\Im y)^2} \zeta(3) \\ &+ \sum_{\vec{b}^2=2} \Re \left[ \tilde{c}(-1) \frac{2y_{-,2}}{\pi} Li_2(e^{2\pi i \frac{\vec{b} \cdot \Im \vec{y}}{y_{-,2}}}) + c(-1) \frac{12y_{-,2}^3}{\pi^3 (\Im y)^2} Li_4(e^{2\pi i \frac{\vec{b} \cdot \Im \vec{y}}{y_{-,2}}}) \right] \\ &+ 4\Re \left\{ \sum_{r>0} \left[ \tilde{c}(kl - \frac{1}{2} \vec{b}^2) Li_1(e^{2\pi i r \hat{y}}) + \frac{6}{\pi (\Im y)^2} c(kl - \frac{1}{2} \vec{b}^2) \mathcal{P}(r \hat{y}) \right] \right\} \end{aligned} \quad (\text{A.42})$$

where we have also introduced the function

$$\mathcal{P}(x) = \Im(x) Li_2(e^{2\pi i x}) + \frac{1}{2\pi} Li_3(e^{2\pi i x}). \quad (\text{A.43})$$

In fact, this integral can be written in the form:

$$\begin{aligned} \tilde{\mathcal{I}}_{8t+2,2}(y) &= 4\Re \left\{ \sum_{r>0} \left[ \tilde{c}(kl - \frac{1}{2} \vec{b}^2) Li_1(e^{2\pi i r \hat{y}}) + \frac{6}{\pi (\Im y)^2} c(kl - \frac{1}{2} \vec{b}^2) \mathcal{P}(r \hat{y}) \right] \right\} \\ &+ \tilde{c}(0) \left( -\log[-y_2]^2 \right) - \mathcal{K} + \frac{1}{(y_2)^2} [\tilde{d}_{ABC}^{8t+2,2} y_2^A y_2^B y_2^C + \delta] \end{aligned} \quad (\text{A.44})$$

where the constant is

$$\delta = \frac{6}{\pi^2} c(0) \zeta(3) \quad (\text{A.45})$$

and  $\tilde{d}$  is a real symmetric tensor, which depends on the Weyl chamber.

We now consider the two special cases  $s = 0$  and  $s = 8$ . For the case  $s = 0$  without loss of generality we have

$$F = \frac{E_4 E_6}{\eta^{24}} \quad (\text{A.46})$$

With  $y_+ = T, y_- = U$  as before we have

$$\begin{aligned}
\tilde{\mathcal{I}}_{2,2}(y) = & 4\Re \left\{ \sum_{r>0} \left[ \tilde{c}(kl) Li_1(e^{2\pi i r \hat{y}}) - \frac{3}{\pi T_2 U_2} c(kl) \mathcal{P}(r \cdot \hat{y}) \right] \right\} \\
& - \frac{\delta}{2T_2 U_2} - 264[-\log[2y_{+,2}y_{-,2}] - \mathcal{K}] \\
& - 48\pi T_2 - 88\pi U_2 + 16\pi \frac{U_2^2}{T_2} \quad T_2 > U_2 \\
& - 48\pi U_2 - 88\pi T_2 + 16\pi \frac{T_2^2}{U_2} \quad U_2 > T_2
\end{aligned} \tag{A.47}$$

As discussed above  $r \cdot \hat{y}$  is only different from  $r \cdot y$  for  $k = 1, \ell = -1$  and in this case it is  $(T_1 - U_1) + i|T_2 - U_2|$ .

When  $s = 8$ , we have

$$F = \frac{E_6}{\eta^{24}} \tag{A.48}$$

To calculate  $\tilde{\mathcal{I}}$  we need

$$\Re \left[ Li_4(e^{i\theta}) \right] = \sum_{j=1}^{\infty} \frac{\cos \theta j}{j^4} = \frac{\pi^4}{90} - \frac{1}{48} \theta^2 (2\pi - \theta)^2 \tag{A.49}$$

valid for  $0 \leq \theta \leq 2\pi$ . We also need the  $E_8$  root sum

$$\sum_{\vec{b}^2=2, \vec{b}>0} (\vec{b} \cdot \vec{v})^4 = 18(\vec{v}^2)^2 \tag{A.50}$$

where the quartic sum can be evaluated using formulae in [106]. Again specializing to the Weyl chamber (5.12) we can write the answer as in (A.44) with the symmetric tensor  $\tilde{d}$  is determined by:

$$\begin{aligned}
\tilde{d}_{ABC}^{10,2} y_2^A y_2^B y_2^C = & -8\pi \left[ (\vec{\rho} \cdot \vec{y}_2 + 41y_{-,2} + 42y_{+,2})(y_2)^2 \right. \\
& \left. - 2 \sum_{\vec{b}^2=2, \vec{b}>0} (\vec{b} \cdot \vec{y}_2)^3 + 60y_{-,2}^2 y_{+,2} + 72y_{+,2}^2 y_{-,2} + 4(y_{-,2})^3 \right]
\end{aligned} \tag{A.51}$$

Finally, we need an explicit formula for the cubic sum over  $E_8$  roots. This is given by

$$\begin{aligned}
\sum_{\vec{b}^2=2, \vec{b}>0} (\vec{b} \cdot \vec{v})^3 = & 6v(1)^2 v(2) + 2v(2)^3 + 6v(1)^2 v(3) + 6v(2)^2 v(3) \\
& + 4v(3)^3 + 6v(1)^2 v(4) + 6v(2)^2 v(4) + 6v(3)^2 v(4) \\
& + 6v(4)^3 + 6v(1)^2 v(5) + 6v(2)^2 v(5) + 6v(3)^2 v(5) \\
& + 6v(4)^2 v(5) + 8v(5)^3 + 6v(1)^2 v(6) + 6v(2)^2 v(6) \\
& + 6v(3)^2 v(6) + 6v(4)^2 v(6) + 6v(5)^2 v(6) + 10v(6)^3 \\
& + 6v(1)^2 v(7) + 6v(2)^2 v(7) + 6v(3)^2 v(7) + 6v(4)^2 v(7) \\
& + 6v(5)^2 v(7) + 6v(6)^2 v(7) + 12v(7)^3 + 30v(1)^2 v(8) \\
& + 30v(2)^2 v(8) + 30v(3)^2 v(8) + 30v(4)^2 v(8) + 30v(5)^2 v(8) \\
& + 30v(6)^2 v(8) + 30v(7)^2 v(8) + 22v(8)^3
\end{aligned} \tag{A.52}$$

where  $v(i)$  is the  $i^{th}$  component of  $v$  in the usual Cartesian basis for  $E_8$  roots.

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